Complex Analysis exam 2023
Solutions

Q 1 (a) $f(x+iy) = \underbrace{2x^4 - 12x^2y^2 + 2y^4 + x^2 - y^2}_{u} + iv(x,y)$

$$u_x = 8x^3 - 24xy^2 + 2x = v_y$$

$$u_y = -24x^2y + 8y^3 - 2y = -v_x$$

So $v = \int u_x \, dy = 8x^3y - 8xy^3 + 2xy + k(x)$

Now $v_x = 24x^2y - 8y^3 + 2y + k'(x)$
$= -u_y$ if $k'(x) = 0$

So $v(x,y) = 8x^3y - 8xy^3 + 2xy + \text{Const}$

$\cancel{f(x}$ $f(z) = 2z^4 + z^2 + \text{const}$

(b) $u_x = v_y$, $u_y = -v_x$

Since $f(x+iy) = u+iv$ is differentiable

Now if $u-iv$ is differentiable
$u_x = -v_y$ must also hold
$u_y = v_x$ simultaneously.

which is only possible if $-v_y = v_y$
$v_x = -v_x$, $u_x = -u_x$, $u_y = -u_y$
ie. $u_x = u_y = v_x = v_y = 0$

So u, v are constants

Q2 (a) $f(z) = z^2 \quad \gamma(t) = e^{it},\ t \in [0,\pi)$

$$\int_\gamma f\,dz = \int_0^\pi (e^{it})^2 \cdot i e^{it}\,dt$$

$$= i\int_0^\pi e^{3it}\,dt$$

$$= \left[\frac{1}{3}e^{3it}\right]_0^\pi = \frac{1}{3}\left(e^{3i\pi} - e^0\right)$$

$$= -\frac{2}{3}.$$

if $t \in [0, 2\pi)$ path is closed, so

$\int_\gamma f = 0$ by Cauchy's Theorem

since f is analytic.

(b) $P = 8y + 3x^4y^5 + y^3, \quad Q = 6xy^3 + 5x^3$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 6y^3 + 15x^2 - (8 + 15x^4y^4 + 3y^2)$$

Thus

$$\oint_C P\,dx + Q\,dy = \int_0^1\int_0^1 (6y^3 + 15x^2 - 8 - 15x^4y^4 - 3y^2)\,dx\,dy$$

$$= \int_0^1 \left[6y^3x + 5x^3 - 8x - 3x^5y^4 - 3xy^2\right]_0^1 dy$$

$$= \int_0^1 (6y^3 + 5 - 8 - 3y^4 - 3y^2)\,dy$$

$$= \frac{6y^4}{4} - 3y - \frac{3y^5}{5} - y^3\Big|_0^1$$

$$= \frac{3}{2} - 3 - \frac{3}{5} - 1 = -\frac{31}{10}.$$

Q3(a) $\int_c \frac{e^{\pi i z}}{z^2-25}dz = \int_c \frac{e^{\pi i z}}{(z-5)(z+5)}dz$

$$= \int \frac{f(z)}{z-5}dz = 2\pi i f(5)$$

$$= \frac{2\pi i e^{5\pi i}}{10} = -\frac{\pi i}{5}$$

$\left(f(z) = \frac{e^{\pi i z}}{z+5}\right)$ By Cauchy integral formula

(b) $f(z) = \frac{z^4 - \cos(z^2)}{z^6}$

$$= \frac{1}{z^2} - \frac{1}{z^6}\cos(z^2)$$

$$= \frac{1}{z^2} - \frac{1}{z^6}\left(1 - \frac{z^4}{2!} + \frac{z^8}{4!} - \frac{z^{12}}{6!} + \right)$$

$$= \frac{1}{z^6} - \left(1 + \frac{1}{2}\right)\frac{1}{z^2} - \frac{z^2}{4!} + \ldots$$

$$= \frac{1}{z^6} - \frac{3/2}{z^2} - \frac{z^2}{4!} +$$

Pole at $z=0$ of order 6

Residue $= 0$.

Q4 (a) $z = e^{i\theta}$ maps $[0, 2\pi)$ to the unit circle C

$\frac{1}{2i}\left(z - \frac{1}{z}\right) = \sin\theta$. $\quad dz = ie^{i\theta}d\theta \rightarrow d\theta = \frac{dz}{iz}$

So $$\int_0^{2\pi} \frac{d\theta}{3+\sin\theta} = \int_C \frac{1}{3+\frac{1}{2i}(z-\frac{1}{z})}\frac{dz}{iz}$$

$$= \int_C \frac{1}{3iz + \frac{1}{2}(z^2-1)}\,dz$$

$$= \int_C \frac{2}{z^2+6iz-1}\,dz = \int_C f\,dz$$

Now $z^2+6iz-1 = 0$ when

$$z = \frac{-6i \pm \sqrt{-36+4}}{2} = (-3 \pm 2\sqrt{2})i$$

$z_1 = (-3+2\sqrt{2})i$ is in unit circle.

$z_2 = (-3-2\sqrt{2})i$ is outside unit circle.

$$\mathrm{Res}(f(z), z_1) = \lim_{z \to (-3+2\sqrt{2})i} \frac{2(z-(-3+\sqrt{2})i)}{z^2+6iz-1}$$

$$= \lim_{z \to (-3+2\sqrt{2})i} \frac{2\,\cancel{(z-(-3+\sqrt{2})i)}}{\cancel{(z-(-3+2\sqrt{2})i)}(z+(3+2\sqrt{2})i)}$$

$$= \frac{2}{-3i+2\sqrt{2}i+3i+2\sqrt{2}i} = \frac{1}{2\sqrt{2}i}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{3+\sin\theta} = 2\pi i\frac{1}{2\sqrt{2}i} = \frac{\pi}{\sqrt{2}}.$$

(b) $I = \int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)(x^2+4)}\,dx$ converges as integrand decays like $\frac{1}{x^3}$.

$f(z) = \frac{ze^{iz}}{(z^2+1)(z^2+4)}$ has poles at $\pm i, \pm 2i$. i and $2i$ are above axis.

$$\mathrm{Res}(f, i) = \lim_{z\to i} \frac{\cancel{(z-i)}ze^{iz}}{(z+i)\cancel{(z-i)}(z^2+4)} = \frac{ie^{-1}}{2i(4-1)} = \frac{e^{-1}}{6}$$

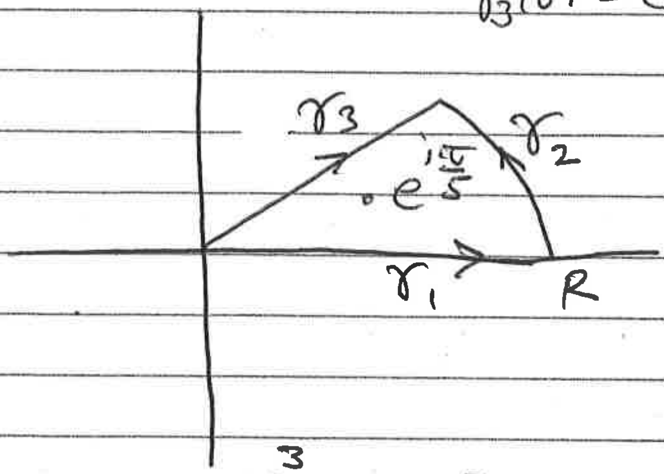
$$\mathrm{Res}(f, 2i) = \lim_{z\to 2i} \frac{\cancel{(z-2i)}ze^{iz}}{(z+2i)\cancel{(z-2i)}(z^2+1)} = \frac{2ie^{-2}}{4i(1-4)} = -\frac{1}{6}e^{-2}.$$

$$\therefore I = \mathrm{Im}\left[\left(\frac{e^{-1}}{6} - \frac{1}{6}e^{-2}\right)2\pi i\right] = \frac{\pi}{3}\left(\frac{1}{e} - \frac{1}{e^2}\right).$$

25. $\int_0^\infty \frac{x^3}{x^5+1}\,dx$ $\quad \gamma_1(t)=t \quad 0\le t\le R$

$\gamma_2(t)=Re^{it} \quad 0\le t\le \frac{2\pi}{5}$

$\gamma_3(t)=e^{\frac{2\pi i}{5}}t \quad 0\le t\le R$



$$f(z)=\frac{z^3}{z^5+1}, \quad \int_\gamma f = 2\pi i \,\mathrm{Res}(f(z), e^{\frac{i\pi}{5}})$$

$$\mathrm{Res}(f(z), e^{\frac{i\pi}{5}}) = \lim_{z\to e^{\frac{i\pi}{5}}} \frac{(z-e^{\frac{i\pi}{5}})z^3}{z^5+1}$$

$$= \lim_{z\to e^{\frac{i\pi}{5}}} \frac{z^3+3z^2(z-e^{\frac{i\pi}{5}})}{5z^4}$$

$$= \frac{1}{5}\frac{1}{(e^{i\pi/5})} = \frac{1}{5}e^{-\frac{i\pi}{5}}.$$ So

$$\int_0^R \frac{t^3}{t^5+1}\,dt + \int_0^{\frac{2\pi}{5}} \frac{R^3e^{3it}\,iRe^{it}}{R^5e^{5it}+1}\,dt$$

$$-\int_0^R \frac{e^{8\pi i/5}t^3}{t^5+1}\,dt = \frac{2\pi}{5}e^{-\frac{i\pi}{5}}$$

Now $\left|\int_0^{2\pi/5} \frac{iR^4e^{4it}}{R^5e^{5it}+1}\,dt\right| \le \int_0^{\frac{2\pi}{5}} \frac{R^4}{|R^5e^{5it}+1|}\,dt$

$$\le \int_0^{2\pi/5} \frac{R^4}{|R^5-1|}\,dt = \frac{2\pi}{5}\frac{R^4}{R^5-1} \to 0$$

as $R\to\infty$

Thus as $R\to\infty$ $\left(1-e^{8\pi i/5}\right)\int_0^\infty \frac{t^3}{t^5+1}\,dt = \frac{2\pi i}{5}e^{-\frac{i\pi}{5}}.$

or $$\int_0^\infty \frac{t^3\,dt}{t^5+1} = \frac{2\pi i}{5}\,\frac{e^{-i\pi/5}}{(1-e^{8\pi i/5})}$$

$$= \frac{2\pi i \; e^{-i\pi/5}}{5\,e^{4\pi i/5}\left(e^{-\frac{4\pi i}{5}}-e^{\frac{4\pi i}{5}}\right)}$$

$$= \frac{2\pi i}{5}\,\frac{e^{-\pi i}}{\left(e^{-\frac{4\pi i}{5}}-e^{4\pi i/5}\right)}$$

$$= \frac{\pi}{5}\,\frac{-2i}{\left(e^{-4\pi i/5}-e^{4\pi i/5}\right)}$$

$$= \frac{\pi}{5\sin\left(\frac{4\pi}{5}\right)}$$

$\left(= \frac{1}{5}\left(2+\frac{2}{\sqrt{5}}\right)^{1/2}\pi \approx 1.06896\right.$

This is not needed but some students may give it)