Question One.

Total marks: 5+5=10.

(a) Let $f(x+iy) = 2x^4 - 12x^2y^2 + 2y^4 - y^2 + iv(x,y)$. Find a function v such that f is differentiable in the whole complex plane? If so, what is v. Express f as a function of a single complex variable z.

(b) Let $f(x+iy) = u(x,y) + iv(x,y)$, be a differentiable function. Show that $g(x+iy) = u(x,y) - iv(x,y)$ cannot be differentiable unless u, v are constants in a region containing $z = x + iy$.

Question Two.

Total marks: 5+5=10.

(a) Let $f(z) = z^2$ and $\gamma(t) = e^{it}, t \in [0, \pi)$. Evaluate $\int_\gamma f(z)dz$. If instead we have $t \in [0, 2\pi)$, what will the value of the integral be? Explain why.

(b) Use Green's Theorem to evaluate the integral

$$\oint_C (8y + 3x^4y^5 + y^3)dx + (6xy^3 + 5x^3)dy$$

where C is the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$ which is traversed counterclockwise.

Question Three.

Total marks: 5+5=10.

(a) Evaluate the contour integral

$$\int_C \frac{e^{\pi i z}}{(z^2-25)}dz,$$

where C is the circle of radius 5 centered at $z=5$, traversed counterclockwise.

(b) Obtain a Laurent series expansion for the function

$$f(z)=\frac{z^4-\cos(z^2)}{z^6},$$

around $z=0$. What is the order of the pole at $z=0$? What is the value of the residue at the pole?

Question Four.

Total marks: 5+5=10.

(a) Use the substitution $z = e^{i\theta}$, $\theta \in [0, 2\pi)$ to show that

$$\int_0^{2\pi} \frac{d\theta}{3 + \sin\theta} = \frac{\pi}{\sqrt{2}}.$$

(b) Using residues, evaluate the integral.

$$\int_{-\infty}^{\infty} \frac{x \sin x dx}{(x^2 + 1)(x^2 + 4)}.$$

Question Five.

Total Marks: 10

(a) Use the contour $\gamma = \gamma_1 + \gamma_2 - \gamma_3$, where

$$\gamma_1(t) = t, \quad 0 \leq t \leq R,$$
$$\gamma_2(t) = Re^{it}, 0 \leq t \leq \frac{2\pi}{5},$$
$$\gamma_3(t) = e^{\frac{2\pi}{5}i}t, 0 \leq t \leq R,$$

to evaluate the integral

$$\int_0^\infty \frac{x^3}{x^5+1} dx.$$

Justify each step.

End of Exam.

Useful information

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1,$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!}f''(z_0)(z - z_0)^2 + \frac{1}{3!}f'''(z_0)(z - z_0)^3 + \cdots$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt, \text{ where } \gamma = \gamma(t),\ t \in [a, b],$$

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi.$$

$$\text{Residue}(f(z), z_0) = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left((z - z_0)^n f(z)\right),$$

for a pole or order n.

$$\text{Residue}(f(z), z) = \lim_{z\to z_0}(z - z_0)f(z),$$

for a pole of order 1.

$$\int_\gamma f(z)dz = 2\pi i\sum_{k=1}^{N}\text{Residue}(f(z), z_k)$$

$z_1, ...z_N$ are the poles of f inside the closed simple curve γ.

$$\sum_{n=-\infty}^{\infty}(-1)^n f(n) = -\sum_{k=1}^{N}\pi\text{Res}\left(\text{cosec}(\pi z)f(z), z_k\right).$$

The Cauchy-Riemann equations are $u_x = v_y$, $u_y = -v_x$.

$$\oint_C Pdx + Qdy = \int\int_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy,$$

where C is the boundary of D.