Complex Analysis Exam 2024 solutions.

Q1(a) $f(x+iy) = u(x,y) + iv(x,y)$

$$v(x,y) = 3x^2y - y^3 + y$$

$$\frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 1$$

Now we require $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

So $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1$

$$u = x^3 - 3xy^2 + x + k(y)$$

$\frac{\partial u}{\partial y} = -6xy + k'(y) = -v_x$ $\therefore k'(y) = 0$
k is a constant

So $u(x,y) + iv(x,y)$

$$= x^3 - 3xy^2 + x + i(3x^2y - y^3 + y) + C$$

$$= z^3 + z + C$$

(b) $u_x = v_y$ and $u_y = -v_x$

Now if $g(x+iy) = 2u(x,y) - 3iv(x,y)$
The Cauchy Riemann equation are not satisfied
Since $2u_x \neq -3v_y$
and $2u_y = 3v_x$
So g is not analytic

Q2(a) $f(z) = z^3$, $\gamma(t) = e^{it}$, $t \in [0, \pi)$

$$\int_\gamma f = \int_0^\pi e^{3it}\, ie^{it} dt = i\int_0^\pi e^{4it}\, dt$$

$$= \frac{1}{4}\left[e^{4it}\right]_0^\pi = \frac{1}{4}\left(e^{4\pi i} - 1\right) = 0$$

, Let $t \in [0, 2\pi)$ we now have a closed curve, integral is zero by Cauchy's Theorem

(b) $$\oint_C (8y^2 + 3x^2y^5 + y^4)\, dx + (6xy^2 + 5x^4)\, dy$$

$$P = 8y^2 + 3x^2y^5 + y^4, \quad Q = 6xy^2 + 5x^4$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 6y^2 + 20x^3 - (16y + 15x^2y^4 + 4y^3)$$

By Green's Theorem

$$\oint_C P\,dx + Q\,dy = \int_0^1\int_0^1 (6y^2 + 20x^3 - 16y - 15x^2y^4 - 4y^3)\, dx\,dy$$

$$= \int_0^1 \left[6xy^2 + 5x^4 - 16xy - 5x^3y^4 - 4y^3x\right]_0^1 dy$$

$$= \int_0^1 (6y^2 + 5 - 16y - 5y^4 - 4y^3)\, dy$$

$$= \left[2y^3 + 5y - 8y^2 - y^5 - y^4\right]_0^1$$

$$= 2 + 5 - 8 - 1 - 1 = -3$$

Q3(a) $\int_c \frac{e^{\pi i z}}{z^2-36} = \int_c \frac{e^{\pi i z}}{(z+6)} \Big/ (z-6)\, dz$

$$f(z) = \frac{e^{\pi i z}}{z+6}$$

By Cauchy integral formula

$$\int_c \frac{f(z)}{z-6}\, dz = 2\pi i\, f(6)$$

$$= \frac{2\pi i}{12} e^{6\pi i z}$$

$$= \frac{\pi i}{6}.$$

(b) $f(z) = \frac{z^2-\cos z}{z^8}$

$$= \frac{1}{z^8}\left(z^2 - \left(1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots\right)\right)$$

$$= \frac{1}{z^8} + \frac{3}{2}\frac{1}{z^6} - \frac{1}{4!\, z^4} + \frac{1}{6!\, z^2} - \frac{1}{8!} + -$$

Poles is of order 8. residue is zero

Q4 (a) $z = e^{i\theta}$, $dz = iz\,d\theta$, C unit circle

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\text{So} \int_0^{2\pi} \frac{1}{4+\cos\theta}\,d\theta = \int_c \frac{1}{4+\frac{1}{2}\left(z+\frac{1}{z}\right)}\,\frac{dz}{iz}$$

$$= \frac{1}{i}\int_c \frac{1}{4z+\frac{1}{2}(z^2+1)}\,dz$$

$$= \frac{1}{i}\int \frac{2}{z^2+8z+1}\,dz$$

poles at $z^2+8z+1=0$, $z_1 = -4+\sqrt{15}$, $z_2 = -4-\sqrt{15}$.

$|z_1| \approx .127 < 1$, $|z_2| > 1$

$$\lim_{z\to z_1} \frac{2(z-z_1)}{(z-z_1)(z-z_2)} = \frac{2}{z_1-z_2} = \frac{1}{\sqrt{15}},$$

Hence

$$\int_0^{2\pi} \frac{1}{4+\cos\theta}\,d\theta = 2\pi i\,\frac{1}{i}\,\frac{1}{\sqrt{15}} = \frac{2\pi}{\sqrt{15}}.$$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{2ix}}{(x^2+4)(x^2+9)}\,dx$ Poles above axis are $\pm 2i$, $\pm 3i$

Function decays faster than $\frac{1}{x^2}$. So we use theorem from lectures

$$\text{Res}\left(\frac{xe^{2ix}}{(x^2+4)(x^2+9)}, 2i\right) = \frac{1}{10}e^{-4}$$

$$\text{Res}\left(\frac{xe^{2ix}}{(x^2+4)(x^2+9)}, 3i\right) = -\frac{1}{10}e^{-6}.$$

So $\displaystyle\int_{-\infty}^{\infty} \frac{x e^{2ix}}{(x^2+4)(x^2+9)} dx = \int_{-\infty}^{\infty} \frac{x(\cos(2x)+i\sin(2x))\,dx}{(x^2+4)(x^2+9)}$

$$= \frac{2\pi i}{10}\left(e^{-4} - e^{-6}\right)$$

Thus

$$\int_{-\infty}^{\infty} \frac{x\sin(2x)}{(x^2+4)(x^2+9)} dx = \frac{\pi}{5}\left(e^{-4}-e^{-6}\right)$$

Hence $\displaystyle\int_{0}^{\infty} \frac{x\sin(2x)}{(x^2+4)(x^2+9)} dx = \frac{\pi}{10}\left(e^{-4}-e^{-6}\right)$

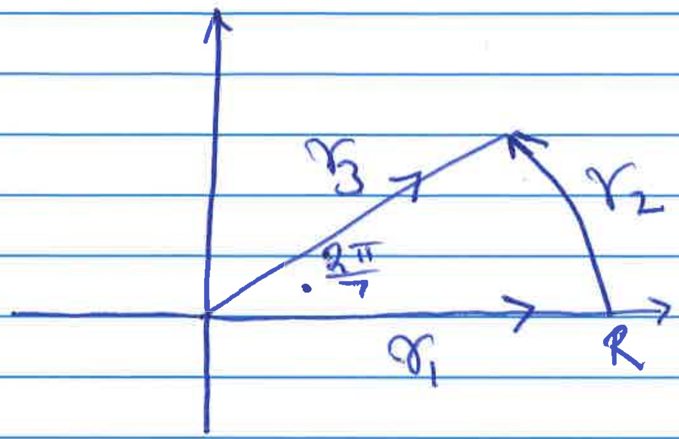
(Q5) $f(z) = \dfrac{z^3}{z^7+1}$.

poles at $z^7+1=0$ or $z^7 = e^{\pi i + 2k\pi i}$

$z_1 = e^{\pi i/7}$, $z_2 = e^{-\pi i/7}$

$z_3 = e^{\frac{3\pi i}{7}}$, $z_4 = e^{-\frac{3\pi i}{7}}$

$z_5 = e^{\frac{5\pi i}{7}}$, $z_6 = e^{\frac{-5\pi i}{7}}$. $z_7 = -1$.



$\gamma_1(t) = t \quad 0 \le t \le R$

$\gamma_2(t) = Re^{it}, \quad t \in [0, \frac{2\pi}{7})$

$\gamma_3 = te^{\frac{2\pi i}{7}}, \quad 0 \le t \le R$

$\gamma = \gamma_1 + \gamma_2 - \gamma_3$

Only pole inside γ is $z = e^{\frac{\pi i}{7}}$.

$$\int_\gamma f(z)\,dz = 2\pi i \operatorname{Res}(f, z_1)$$

$$\operatorname{Res}(f(z), z_1) = \lim_{z\to z_1} \frac{(z-z_1)z^3}{z^7+1}$$

$$= \lim_{z\to z_1} \frac{z^3 + 3(z-z_1)z^2}{7z^6}$$

$$= \frac{e^{\frac{3\pi i}{7}}}{7e^{6\pi i/7}} = \frac{1}{7} e^{-\frac{3\pi i}{7}}.$$

$$\int_{\gamma_1} f = \int_0^R \frac{t^3}{t^7+1}\,dt, \quad \int_{\gamma_2} f = \int_0^{\frac{2\pi}{7}} \frac{R^3 e^{3it}}{R^7 e^{7it}+1} Re^{it}\,dt$$

$$\int_{\gamma_3} f = \int_0^R \frac{t^3 e^{\frac{6\pi i}{7}}}{t^7+1} e^{\frac{2\pi i}{7}}\,dt = e^{\frac{8\pi i}{7}} \int_0^R \frac{t^3}{t^7+1}\,dt$$

Now $\left|\int_0^{\frac{2\pi}{7}} \frac{R^4 e^{4it}}{R^7 e^{7it}+1}\,dt\right| \le \int_0^{\frac{2\pi}{7}} \frac{R^4}{|R^7 e^{7it}+1|}\,dt$

$$\le \int_0^{\frac{2\pi}{7}} \frac{R^4}{R^7-1}\,dt = \frac{R^4}{R^7-1}\,\frac{2\pi}{7} \longrightarrow 0$$

as $R \longrightarrow \infty$.

So $\int_\gamma f = \int_0^R \frac{t^3}{t^7+1}\,dt + \int_{\gamma_2} - e^{\frac{8\pi i}{7}} \int_0^R \frac{t^3}{t^7+1}\,dt$

$$= 2\pi i\, e^{-\frac{3\pi i}{7}}$$

Let $R \to \infty$

$$\int_0^\infty \frac{t^3}{t^7+1}\,dt = \frac{1}{7}\,\frac{2\pi i\, e^{-\frac{3\pi i}{7}}}{1-e^{8\pi i/7}}$$

$$= \frac{2\pi i\, e^{-\frac{3\pi i}{7}}}{e^{4\pi i/7}\left(e^{-\frac{4\pi i}{7}} - e^{4\pi i/7}\right)}\,\frac{1}{7}$$

$$= \frac{2\pi i\, e^{-\pi i}}{7\left(e^{-\frac{4\pi i}{7}} - e^{4\pi i/7}\right)} = \frac{-2\pi i}{e^{-\frac{4\pi i}{7}} - e^{4\pi i/7}}\,\frac{1}{7}$$

$$= \frac{\pi}{7\sin\left(\frac{4\pi}{7}\right)}.$$