Complex Analysis - 24557656

Zachary Zerafa

1)

a) $f(x+iy) = (x^4 - 6x^2y^2 + y^4 + x) + i\,v(x,y)$ $\qquad u_x = v_y$, $u_y = -v_x$

f is differentiable $\Rightarrow 4x^3 - 12xy^2 + 1 = v_y$

$\Rightarrow v(x,y) = 4x^3y - 4xy^3 + y + h(x)$

$\because u_y = -v_x \Rightarrow -12x^2y + 4y^3 - h'(x) = -12x^2y + 4y^3$

$\Rightarrow h'(x) = 0 \Rightarrow h(x) = c$

$\Rightarrow \underline{\underline{v(x,y) = 4x^3y - 4xy^3 + y + c}}$

$\underline{\underline{f(z) = z^4 + z + ci}}$

$(x+iy)^4 = 1x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + 1y^4$

$= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$

b) $f(x+iy) = u(x,y) + iv(x,y)$ is ~~differentiable everywhere~~ ~~Holomorphic~~ Entire

$h(x+iy) = (x+iy)(u(x,y) + iv(x,y))$

$= (xu(x,y) - yv(x,y)) + i(xv(x,y) + yu(x,y))$

$\qquad h_u \qquad\qquad\qquad h_v$

h is entire $\Leftrightarrow \forall x+iy \in \mathbb{C}\,[h_{u_x} = h_{v_y} \wedge h_{u_y} = -h_{v_x} \wedge$ differentiably continuous derivatives$]$

$h_{u_x} = \cancel{(xu)' - (yv)'} = u + x\frac{\partial u}{\partial x} - y\frac{\partial v}{\partial x}$ $\qquad h_{u_y} = x\frac{\partial u}{\partial y} - v - y\frac{\partial v}{\partial y}$

$h_{v_x} = v + x\frac{\partial v}{\partial x} + y\frac{\partial u}{\partial x}$ $\qquad h_{v_y} = x\frac{\partial v}{\partial y} + u + y\frac{\partial u}{\partial y}$

Note that since f is differentiable... $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

Next Page...

$$\therefore \quad h_{u_x} = u + x\frac{\partial v}{\partial y} + y\frac{\partial u}{\partial y} = h_{v_y}$$

Cauchy Riemann satisfied!

$$h_{u_y} = -x\frac{\partial v}{\partial x} - v - y\frac{\partial u}{\partial x} = -h_{v_x}$$

~~functions are conti~~ continuity is closed under multiplication,

$\therefore$ $h(z)$ is Entire!

2) $f(z) = z^4 + z^2 \qquad \gamma(t) = e^{it} \qquad t \in [0, \pi]$

$$I = \int_\gamma f(z)\,dz$$

$$= \int_0^\pi \left(e^{i4t} + e^{i2t}\right) i e^{it}\,dt$$

$$= i\int_0^\pi e^{i5t} + e^{i3t}\,dt$$

$$= i\left[\frac{e^{i5t}}{5i} + \frac{e^{i3t}}{3i}\right]_0^\pi$$

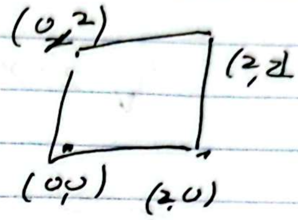
$$= i(0) = 0$$

~~ff~~ $\int_\gamma z^4 + z^2\,dz = 0$ where $\gamma(t) = e^{it}$ $t \in [0, 2\pi]$

due to Cauchy's theorem; $z^4 + z^2$ is entire, so any closed integral ~~integral~~ on a simply connected set results to 0.

$z^4 + z^2$ has no singularities, hence $\int_\gamma f(z)\,dz = 0$.

b) $I = \oint_C (8y^2+3x^2y^4+y^3+\sin x)\,dx+(6x^2y^2+5x^3+\cos y)\,dy$



$$I=\iint_D (12xy^2+15x^2-16y-12x^2y^3-3y^2)\,dx\,dy$$

$$=\int_0^2\int_0^2 (12xy^2+15x^2-16y-12x^2y^3-3y^2)\,dx\,dy$$

$$=\int_0^2 \left[6x^2y^2+5x^3-16yx-4x^3y^3-3y^2x\right]_0^2 dy$$

$$=\int_0^2 24y^2+40-32y-32y^3-6y^2\,dy$$

$$=\int_0^2 \left[8y^3+40y-16y^2-8y^4-2y^3\right]_0^2 dy$$

$$=64+80-64-128-16$$

$$=\underline{\underline{-64}}$$

3)

a) $I = \oint_C \frac{e^{\pi z}}{z^2+36} dz$

$$I = \frac{1}{2\pi i} \oint_C \frac{2\pi i e^{\pi z}/(z-6i)}{(z+6i)} dz$$

$$\therefore \quad I = 2\pi i \frac{e^{\pi(-6i)}}{((-6i)-6i)} = \frac{2\pi i}{-12i} = -\frac{\pi}{6}$$

b) $f(z) = z^{-5} - \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{z} z^{2n+1-8}}{(2n+1)!}$

$$= z^{-5} - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-7}}{(2n+1)!}$$

$$= z^{-5} - z^{-7} + \frac{z^{-5}}{3!} - \frac{z^{-3}}{5!} + \frac{z^{-1}}{7!} - \sum_{n=4}^{\infty} \frac{(-1) z^{2n-7}}{(2n+1)!}$$

$\therefore$ 0 is a pole of order 7

$\wedge$ $\operatorname{Res}(f,0) = \frac{1}{7!} = \frac{1}{5040}$

CamScannerでスキャン

4)

a) $I = \int_0^{2\pi} \frac{d\theta}{4-\cos\theta}$ $z = e^{i\theta}$

$I = \oint_\gamma \frac{1}{4 - \frac{z+\frac{1}{z}}{2}} \frac{dz}{iz}$ ~~Tcis~~ $\gamma(t) = e^{it}$ $t \in [0, 2\pi]$

$= \oint_\gamma \frac{dz}{(8z - z^2 - 1)iz}$ $z^2 - 8z + 1 = 0$

$\Rightarrow z = \frac{8 \pm \sqrt{64-4}}{2} = 4 \pm \frac{\sqrt{60}}{2}$

$= 4 \pm \sqrt{15}$

$= i \oint_\gamma \frac{dz}{-z(z - 4 + \sqrt{15})(z - 4 - \sqrt{15})}$

$= i \oint_\gamma \frac{dz}{z(z-4+\sqrt{15})(z-4-\sqrt{15})}$ 0 and $4-\sqrt{15}$ are poles in γ's interior !

$= i\, 2\pi i \left(\mathrm{Res}(f,0) + \mathrm{Res}(f, 4-\sqrt{15})\right)$

$= -2\pi \left(\frac{1}{(-4+\sqrt{15})(-4-\sqrt{15})} + \frac{1}{(4-\sqrt{15})(-2\sqrt{15})} \right)$

$= -2\pi \left(\frac{(4-\sqrt{15})(-2\sqrt{15}) + (-4+\sqrt{15})(-4-\sqrt{15})}{(4-\sqrt{15})^2(-4-\sqrt{15})2\sqrt{15}} \right)$

$= -2\pi \left(\frac{(-2\sqrt{15} + 4 + \sqrt{15})}{(4-\sqrt{15})^2(-4-\sqrt{15})2\sqrt{15}} \right)$

$= -2\pi \left(\frac{1}{(4-\sqrt{15})(4-\sqrt{15})2\sqrt{15}} \right)$

$= -2\pi \left(-\frac{1}{\sqrt{15}} \right)$

$= \frac{2\pi}{\sqrt{15}}$

4)

b) $I=\int_0^\infty \frac{\cos x}{(x^2+4)(x^2+1)}\,dx$

$f(z)=\frac{e^{iz}}{(z^2+4)(z^2+1)}$

$\gamma_1(t)=t$
$\gamma_2(t)=Re^{it}$
$\gamma_3(t)=-t$

$$J=\lim_{R\to\infty}\int_\gamma f(z)\,dz$$
$$=2\pi i\left(\mathrm{Res}(f,2i)+\mathrm{Res}(f,i)\right)$$
$$=2\pi i\left(\lim_{z\to 2i}\frac{e^{iz}}{(z+2i)(z^2+1)}+\lim_{z\to i}\frac{e^{iz}}{(z+i)(z^2+4)}\right)$$
$$=2\pi i\left(\frac{e^{-2}}{4i(-3)}+\frac{e^{-1}}{2i(3)}\right)$$
$$=2\pi i\left(\frac{-e^{-2}}{12i}+\frac{2e^{-1}}{12i}\right)$$
$$\boxed{J=\frac{\pi(2e^{-1}-e^{-2})}{6}}$$

$$\sqrt{(R^4\cos 4t+R^2\cos 2t+4)^2+(R^4\sin 4t+\cancel{5R^2}\sin 2t)^2}$$
$$=\sqrt{R^8\cos^2 4t+2R^6\cos 4t\cos 2t+8R^4\cos 4t+R^4\cos^2 2t+16+8R^2\cos 2t+R^8\sin^2 4t+25R^4\sin^2 2t+10R^6\sin 4t\sin 2t}$$
$$\geq\sqrt{R^8+16}$$

order 4 Polynomial

$$J=\int_{\gamma_1}f+\int_{\gamma_2}f-\int_{\gamma_3}f$$

$$\boxed{\int_{\gamma_1}f=I}\qquad\boxed{\int_{\gamma_3}f=-I}$$

$$\left|\int_{\gamma_2}f\right|\leq\int_0^\pi\left|\frac{e^{iRe^{it}}\,Rie^{it}}{(R^2e^{2it}+4)(R^2e^{2it}+1)}\right|dt\leq\int_0^\pi\frac{R^2}{|R^4e^{4it}+5R^2e^{2it}+4|}\,dt\leq\int_0^\pi\frac{R^2\,dt}{\text{order 4 Polynomial}}$$
$$\to 0$$

$$\therefore\ \frac{\pi(2e^{-1}-e^{-2})}{6}=2I$$

$$\boxed{I=\frac{\pi(2e^{-1}-e^{-2})}{12}}$$

5)

a) $I = \int_0^\infty \frac{x}{x^7+1}\,dx$

$\gamma = \gamma_1 + \gamma_2 - \gamma_3$

$\gamma_1(t) = t$

$\gamma_2(t) = Re^{it}$

$\gamma_3(t) = te^{\frac{2\pi}{7}i}$

$C = \{e^{i\frac{3\pi}{7}}, e^{-i\frac{3\pi}{7}}, e^{-i\frac{5\pi}{7}},$
$e^{i\frac{5\pi}{7}}, e^{i\frac{\pi}{7}}, e^{-i\frac{\pi}{7}}, -1\}$

$\boxed{*}$

$J = \lim_{R\to\infty} \int_\gamma f(z)\,dz$

$= 2\pi i\left(\mathrm{Res}(f, e^{i\frac{\pi}{7}})\right)$

$= 2\pi i\left(\lim_{z\to e^{i\frac{\pi}{7}}} \frac{2z - e^{i\frac{\pi}{7}}}{7z^6}\right)$

$= 2\pi i \frac{2e^{i\frac{\pi}{7}} - e^{i\frac{\pi}{7}}}{7e^{i\frac{6\pi}{7}}}$

$\boxed{= \frac{2\pi i}{7} e^{-i\frac{5\pi}{7}}}$

$I = \frac{2\pi i e^{-i\frac{5\pi}{7}}}{7(1 - e^{i\frac{4\pi}{7}})}$

$= \frac{2\pi i}{7(e^{i\frac{5\pi}{7}} - e^{i\frac{9\pi}{7}})}$

$\cancel{= \frac{2\pi i}{7(e^{i\frac{5\pi}{7}} - e^{-i\frac{5\pi}{7}})} = \frac{\pi}{7\sin(\frac{2}{7}\pi)}}$

$J = \int_{\gamma_1} f + \int_{\gamma_2} f - \int_{\gamma_3} f$

$I = \frac{2\pi i e^{-i\frac{14}{7}\pi}}{7(e^{-i\frac{9\pi}{7}} - e^{-i\frac{5\pi}{7}})}$

$\boxed{\int_{\gamma_1} f = I}$ $\boxed{\int_{\gamma_3} f = e^{i\frac{4\pi}{7}} I}$

$\boxed{I = \frac{\pi}{7\sin(\frac{2\pi}{7})}}$

$\left|\int_{\gamma_2} f\right| \le \int_0^{\frac{2\pi}{7}} \frac{R\,dt}{|R e^{7it}+1|} \le \int_0^{\frac{2\pi}{7}} \frac{R\,dt}{|R^7-1|}$

$= \frac{2\pi}{7}\frac{R}{|R^7-1|} \to 0$

$\therefore \boxed{\frac{2\pi i}{7} e^{-i\frac{5\pi}{7}} = (1 - e^{i\frac{4\pi}{7}}) I}$ continued at $*$