Complex Analysis Class Test One and Two

Time Allowed: 60 minutes.

Nonprogrammable calculators may be used.

(1) Use the Cauchy Riemann equations to prove that

$$f(z) = z^3 + 3z$$

is differentiable everywhere.

(2) Evaluate the line integral

$$\oint_C (x^4 + 3y^2x)dx + (2x^2y - y^3x)dy,$$

where C is the square with vertices $(0,0)$, $(3,0)$, $(3,2)$ and $(0,2)$, traversed counterclockwise.

(3) Evaluate the contour integral

$$I = \int_\gamma \frac{e^{\pi z}}{z^2+4} dz$$

where γ is the circle of radius 2 centered at $2i$.

(4) Use the substitution $z = e^{it}$ to show that

$$\int_0^{2\pi} (\cos^3 t + \sin^2 t)dt = \pi.$$

(5) Use the residue theorem to evaluate the integral

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)} dx.$$

Formulas Over Page

Useful information

$$e^{iz} = \cos z + i \sin z.$$

$$u_x = v_y, u_y = -v_x$$

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt, \text{ where } \gamma = \gamma(t),\ t \in [a,b],$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

$$\text{Residue}(f(z), z) = \lim_{z \to z_0} (z - z_0) f(z),$$

for a pole of order 1.

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \text{Residue}(f(z), z_k)$$

$z_1, \ldots z_N$ are the poles of f inside the closed simple curve γ.

$$\oint_C P dx + Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy,$$

where C is the boundary of D.

CA test 2025 Solutions

(1) $f(z) = z^3 + 3z, \quad z = x+iy$

$= (x+iy)^3 + 3(x+iy)$

$= x^3 + 3ix^2y - 3xy^2 - iy^3 + 3x + 3iy$

$= u + iv$

$u(x,y) = x^3 - 3xy^2 + 3x, \quad v = 3x^2y - y^3 + 3y$

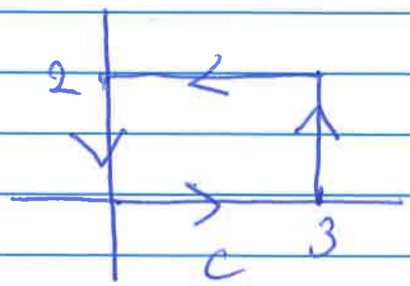
$u_x = 3x^2 - 3y^2 + 3, \quad v_y = 3x^2 - 3y^2 + 3$

$\therefore u_x = v_y$

$u_y = -6xy, \quad v_x = 6xy \quad \therefore u_y = -v_x$

These hold for all x, y. Therefore f is differentiable everywhere

(2) $\int_C (x^4 + 3y^2x)\,dx + (2x^2y - y^3x)\,dx = \int_C P\,dx + Q\,dy$



$$= \int_0^3 \int_0^2 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$$

$$= \int_0^3 \int_0^2 (4xy - y^3 - 6yx)\,dy\,dx$$

$$= -\int_0^3 \int_0^2 (y^3 + 2xy)\,dy\,dx$$

$$= -\int_0^3 \left[y^4/4 + xy^2\right]_0^2 dx$$

$$= -\int_0^3 (4 + 4x)\,dx = -\left[4x + 2x^2\right]_0^3$$

$$= -(12 + 18) = -30$$

(3) $$\int_\gamma \frac{e^{\pi z}}{z^2+4}\,dz = \int_\gamma \frac{e^{\pi z}}{(z+2i)} \Big/ (z-2i)\,dz$$

$$= 2\pi i f(2i), \quad f(z) = \frac{e^{\pi z}}{z+2i}$$

$$= 2\pi i \frac{e^{2\pi i}}{4i} = \frac{\pi}{2}$$

(2)

(4) $z = e^{it}$. $\cos t = \frac{1}{2}(e^{it} + e^{-it}) = \frac{1}{2}(z + \frac{1}{z})$

$\sin t = \frac{1}{2i}(z - \frac{1}{z})$. $t \in [0, 2\pi]$ maps to the unit circle C

$$\cos^3 t + \sin^2 t = \frac{1}{2^3}\left(z+\frac{1}{z}\right)^3 - \frac{1}{4}\left(z-\frac{1}{z}\right)^2$$
$$= \frac{1}{8}\left(z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}\right) - \frac{1}{4}\left(z^2 - 2 + \frac{1}{z^2}\right)$$

$\frac{dz}{iz} = dt$.

Thus $$\int_0^{2\pi}(\cos^3 t + \sin^2 t)\,dt = \int_C \left[\frac{1}{8}\left(z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}\right) - \frac{1}{4}z^2 + \frac{1}{2} - \frac{1}{4z^2}\right]\frac{dz}{iz}$$
$$= \int_C \underbrace{\left(\frac{z^2}{8i} + \frac{3}{8i} + \frac{3}{z^2 i} + \frac{1}{z^4 i} - \frac{1}{4i}z + \frac{1}{2iz} - \frac{1}{4iz^3}\right)}_{f(z)} dz$$

f has a pole at $z = 0$, which is inside C

$$\mathrm{Res}(f, 0) = \frac{1}{2i}$$

$$\therefore \int_0^{2\pi}(\cos^3 t + \sin^2 t)\,dt = 2\pi i \cdot \frac{1}{2i} = \pi.$$

(5) $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ has simple poles at $z = \pm i$, $z = \pm 2i$

f decays faster than $1/z^2$ so we can use theorem from lectures. We need poles above axis

$$\mathrm{Res}(f, i) = \lim_{z \to i} \frac{z^2 \cancel{(z-i)}}{\cancel{(z-i)}(z+i)(z^2+4)} = \frac{-1}{6i}$$

$$\mathrm{Res}(f, 2i) = \lim_{z \to 2i} \frac{\cancel{(z-2i)}\, z^2}{\cancel{(z-2i)}(z+2i)(z^2+1)} = \frac{-4}{-12i}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = 2\pi i\left(\frac{1}{3i} - \frac{1}{6i}\right)$$

$$= \frac{\pi}{3}.$$

Thus $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}$