Question One.

Total marks: 5+5=10.

(a) Let $f(x + iy) = x^4 - 6x^2y^2 + y^4 + x + iv(x, y)$. Can you find a function v such that f is differentiable in the whole complex plane? If so, what is v? Express f as a function of a single complex variable z.

(b) Let $f(x+iy) = u(x, y)+iv(x, y)$, be a function which if differentiable everywhere. Prove that $h(z) = zf(z)$ is also differentiable everywhere.

Question Two.

Total marks: 5+5=10.

(a) Let $f(z) = z^4 + z^2$ and $\gamma(t) = e^{it}, t \in [0, \pi)$. Evaluate $\int_\gamma f(z)dz$. If instead we have $t \in [0, 2\pi)$, what will the value of the integral be? Explain why without explicitly evaluating the contour integral.

(b) Use Green's Theorem to evaluate the line integral

$$\oint_C (8y^2 + 3x^2y^4 + y^3 + \sin x)dx + (6x^2y^2 + 5x^3 + \cos y)dy$$

where C is the square with vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$ which is traversed counterclockwise.

Question Three.

Total marks: 5+5=20.

(a) Evaluate the contour integral

$$\int_C \frac{e^{\pi z}}{(z^2+36)} dz,$$

where C is the circle of radius 6 centered at $z = -6i$, traversed counterclockwise.

(b) Obtain a Laurent series expansion for the function

$$f(z) = \frac{z^3 - \sin z}{z^8},$$

around $z = 0$. What is the order of the pole at $z = 0$? What is the value of the residue at the pole?

Question Four.

Total marks: 5+5=10.

(a) Use the substitution $z = e^{i\theta}$, $\theta \in [0, 2\pi)$ to show that
$$\int_0^{2\pi} \frac{d\theta}{4 - cos\theta} = \frac{2\pi}{\sqrt{15}}.$$

(b) Using residues, evaluate the integral.
$$\int_0^{\infty} \frac{\cos x}{(x^2+4)(x^2+1)} dx.$$
You may use the relevant theorem from lectures.

Question Five.

Total Marks: 10

(a) Use the contour $\gamma = \gamma_1 + \gamma_2 - \gamma_3$, where

$$\gamma_1(t) = t, \qquad 0 \leq t \leq R,$$
$$\gamma_2(t) = Re^{it}, 0 \leq t \leq \frac{2\pi}{7},$$
$$\gamma_3(t) = te^{\frac{2\pi}{7}i}, 0 \leq t \leq R,$$

together with the Cauchy Residue Theorem to evaluate the integral

$$\int_0^\infty \frac{x}{x^7+1} dx.$$

Justify each step.

End of Exam

Useful information

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1,$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!}f''(z_0)(z - z_0)^2 + \frac{1}{3!}f'''(z_0)(z - z_0)^3 + \cdots$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt, \text{ where } \gamma = \gamma(t),\ t \in [a, b],$$

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi.$$

$$\text{Residue}(f(z), z_0) = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left((z - z_0)^n f(z)\right),$$

for a pole or order n.

$$\text{Residue}(f(z), z) = \lim_{z\to z_0}(z - z_0)f(z),$$

for a pole of order 1.

$$\int_\gamma f(z)dz = 2\pi i\sum_{k=1}^{N}\text{Residue}(f(z), z_k)$$

$z_1, \ldots z_N$ are the poles of f inside the closed simple curve γ.

$$\sum_{n=-\infty}^{\infty}(-1)^n f(n) = -\sum_{k=1}^{N}\pi\text{Res}\left(\text{cosec}(\pi z)f(z), z_k\right).$$

The Cauchy-Riemann equations are $u_x = v_y$, $u_y = -v_x$.

$$\oint_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy,$$

where C is the boundary of D.