Calculus – The basics

Complex Analysis is essentially about calculus for functions of a complex variable.

So the basic set we are interested in is $\mathbb{C}$. To recall the basics, and we will say more about this, the complex numbers are

$$\mathbb{C} = \{a+ib,\ a,b \in \mathbb{R},\ i=\sqrt{-1}\}.$$

$\mathbb{R}$ denotes the real numbers. We also remind you that

$\mathbb{N} = \{1,2,3,\dots\}$ the natural numbers. we can also include 0 in $\mathbb{N}$ if we wish. For most purposes it makes no differen[ce]

$\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \dots\}$ The integers

(Why $\mathbb{Z}$? From Zahlen - number in German)

$\mathbb{Q}$ = rational numbers. ie numbers of the form $\frac{a}{b}$, $a,b \in \mathbb{Z}$.

So $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

'$\subset$' means 'contained in'. $A \subset B$ means A is a subset of B.

Calculus was first developed for functions of real numbers.

Recall that a function f takes input from some set X and returns a unique value in a set Y.

We write this as $f: X \to Y$.
In real variable calculus $X \subseteq \mathbb{R}$, $Y \subseteq \mathbb{R}$
The symbol $\subseteq$ means contained in or equal to. So X, Y might be the whole of $\mathbb{R}$.

X is the <u>domain</u>

Y is the <u>Range</u>

eg. $f(x) = x^2$, Let $X = \mathbb{R}$.
Then $Y = \mathbb{R}_0^+ = \{x \in \mathbb{R}, x \geq 0\}$. Since $x^2 \geq 0$ for all $x \in \mathbb{R}$.
The subscript $_0$ just means that 0 is included.
The symbol $\in$ means 'element of'.

We often write $\mathbb{R}^+ = \{x \in \mathbb{R}, x > 0\}$.

In calculus we are usually concerned with special classses of functions.
So we have the notion of a continuous function. There is an old notion that a continuous function is one that can be drawn without taking your pencil off the page
This is not actually true, nor is it useful.

A better idea is that a function f is continuous if $f(x)$ and $f(y)$ are close together if x and y are close together.

which means that $|f(x)-f(y)|$ is small if $|x-y|$ is small. Fix an $x \in X$.
Here $|a|$ is the absolute value defined by
$$|a| = \sqrt{a^2}$$
So $|-3| = 3$, $|6| = 6$ etc

To return to the idea of continuity we can refine our idea as follows.
f is going to be continuous if we can make $|f(x)-f(y)|$ as small as we like by specifying how close y has to be to x. That is by saying how small $|x-y|$ must be.

So if we want $|f(x)-f(y)| < \frac{1}{100}$, say we might require $|x-y| < \frac{1}{1000}$
In this case, if $x, y \in X$ and $|x-y| < \frac{1}{1000}$, then $|f(x)-f(y)| < \frac{1}{100}$
There is nothing special about these numbers.
So we choose an arbitrary number ε (epsilon — a Greek e)
We want $|f(x)-f(y)| < \varepsilon$.
How small does $|x-y|$ have to be to make this true? If we can find a δ (Delta — Greek d) such that if
$$|x-y| < \delta$$
then $|f(x)-f(y)| < \varepsilon$,

then we say f is continuous at x.

The points y can be thought of as forming an interval around x
If we can do this for all $x \in X$, we say that f is continuous on X.

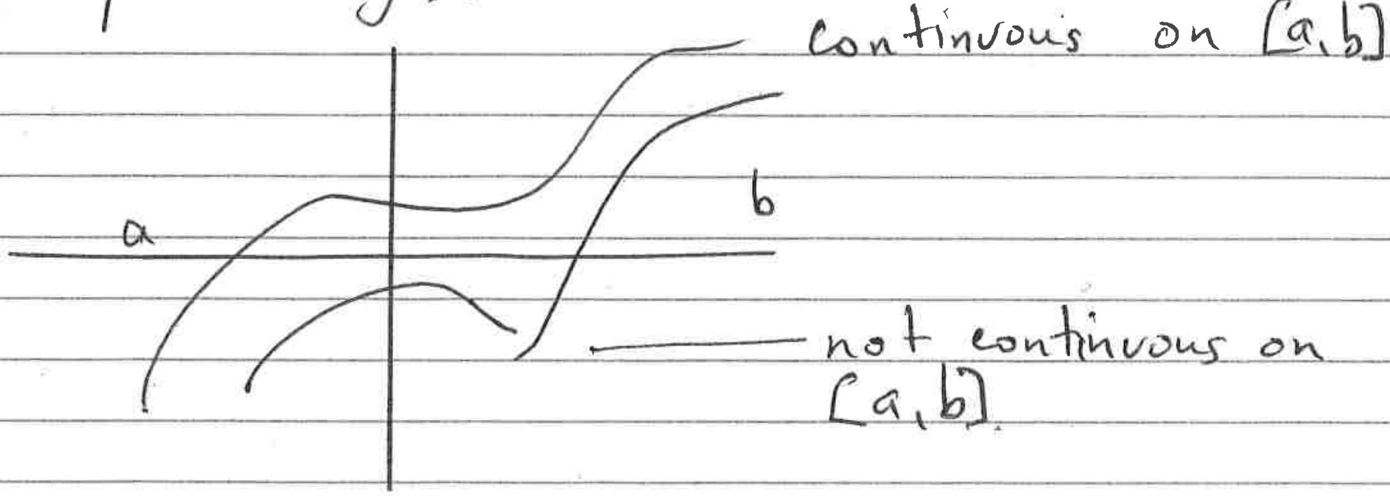
Continuous functions are special. They have many important and highly useful functions.
Here are three
(1) If $f: [a,b] \to \mathbb{R}$, a, b are finite then f attains its maximum and minimum values on $[a,b]$.
(2) $f: [a,b] \to \mathbb{R}$ is uniformly continuous (we will not discuss this).
(3) f has the intermediate value property
If $f: [a,b) \to \mathbb{R}$

$$f(a) < 0, \quad f(b) > 0$$

then there is a $c \in (a,b)$ such that

$$f(c) = 0$$

Continuous functions are easy to recognise for most purposes. (There are some strange examples though).



Most importantly continuous functions can be integrated. So if f is continuous on $[a,b]$ $\int_a^b f(x)dx$ exists. We will come back to integration.

The first major part of calculus is differentiation.

We suppose that f is a function
First we define a limit

We say $f(x) \to L$ as $x \to a$

($\to$ means gets closer and closer to)

if given $\varepsilon > 0$ we can find $\delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$.
Now we define the derivative.
We say that f is differentiable at x if

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

exists.
f' is called the derivative of f.
We also write

$$\frac{df}{dx} = f'(x).$$

We can define partial derivatives for functions of several variables.

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{k \to 0} \frac{f(x,y+k)-f(x,y)}{k}.$$

<u>Example</u> $f(x,y) = x^2 y$.

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{(x+h)^2 y - x^2 y}{h}$$

$$= y \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 - x^2)}{h}$$

$$= y \lim_{h\to 0} (2x+h) = 2xy.$$

The derivative is one of the most powerful tools in mathematics, but we will not discuss its properties here. See your Math1 notes.

We will be concerned about what the derivative means in the complex plane. So if $z \in \mathbb{C}$, what does $f'(z)$ mean???

For continuous functions f we can define the integral using Riemann sums. We will not go into this here as it is quite involved. The most important result in mathematics is the Fundamental Theorem of Calculus.

If f is continuous

(i) $$\frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

(ii) If $F' = f$ $$\int_a^b f(t)\,dt = F(b) - F(a).$$

Complex Analysis is primarily concerned with integration in the complex plane.

What does this even mean? We think of $\int_a^b f(t)dt$ as the area under f between a, b, (if $f > 0$)

What would $\int_a^b f(z)dz$, $a, b \in \mathbb{C}$

mean? We will learn all about this in the coming weeks.