Complex Analysis - Workshop 1
Revision

It is often thought that complex numbers arose from the problem of solving equations of the form $x^2+1=0$.

In fact this is false. Complex numbers first appeared when people tried to solve cubic equations. This took place at the end of the 15th century and progress was made well into the 16th century. The notes have the story in more detail

By the time of Gauss, complex numbers were becoming accepted. The question then arose as to how calculus works with complex numbers

So if $z = x+iy$, $x, y \in \mathbb{R}$, $i=\sqrt{-1}$
The set $\mathbb{C}$ denotes the complex numbers
what does it mean to take the derivative of $f(z)$ with respect to z? To answer this requires analysis
So let us step back to the real case.

We begin with sequences of real numbers $\{a_n\}$
We say that a sequence a_n converges to a if we can make the quantity $|a_n - a|$ as small as we like by choosing n large enough.

To make this precise, let $\varepsilon > 0$ be a real number. We say that $a_n \to a$

if given any $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that if $n \geq N$ then

$$|a_n - a| < \varepsilon.$$

If $a_n \to a$ (converges to) then we say that the sequence $\{a_n\}_{n=1}^{\infty}$ is <u>convergent</u>.

What about complex numbers?

We define $|z| = \sqrt{x^2+y^2}$, where $z = x+iy$. $|z|$ is called the modulus of z.

What we can do is define convergence the same way.

<u>Definition</u> Let $\{z_n\}$ be a sequence of complex numbers. We say that z_n converges to z if given $\varepsilon > 0$, we can find $N \in \mathbb{N}$, such that

$$n \geq N \Rightarrow |z_n - z| < \varepsilon.$$

Sequences of complex numbers have mostly <u>the same properties as sequences of real numbers</u>

(1) If $z_n \to z$, $\lambda z_n \to \lambda z$, $\lambda \in \mathbb{C}$

(2) $z_n \to z$, $w_n \to w$ then $z_n + w_n \to z + w$.

(iii) $z_n \to z$, $w_n \to w$, then $z_n w_n \to zw$

(iv) $z_n \to z$, $w_n \to w$, $w_n \neq 0$, $w \neq 0$ then
$$\frac{z_n}{w_n} \to \frac{z}{w}.$$

Similar properties hold for series

If $\{a_n\}$ is a sequence of real numbers the series
$$\sum_{n=1}^{\infty} a_n$$
is convergent if the sequence
$$S_N = \sum_{n=1}^{N} a_n$$
is convergent

For complex numbers the same definition works
If $\{z_n\}$ is a sequence of complex numbers, the series
$$\sum_{n=1}^{\infty} z_n$$
converges if and only if the sequence
$$S_N = \sum_{n=1}^{N} z_n$$
converges.

The same tests work. For example consider

$$\sum_{n=1}^{\infty} z_n$$

This is convergent if

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| < 1.$$

It is divergent if

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| > 1.$$

If $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 1$, then the test is inconclusive.

This is called the ratio test.

Much of the theory of real analysis goes over to complex analysis unchanged.

The most important operations in analysis are differentiation and integration.

We will not talk about integration here but much of the subject is devoted to it.

What about the derivative?

Let $f : \mathbb{R} \to \mathbb{R}$.

We define the derivative of f as

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \qquad (1)$$

provided the limit exists.

We also write $f'(x) = \frac{df}{dx}$.

Now we can define the derivative of $f : \mathbb{C} \to \mathbb{C}$ in exactly the same way.

So we would have

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad (2)$$

This is basically the same thing.

But is it really?

Question to think about.
In what way do (1) and (2) differ? Because they do and the difference is profound

We will discuss this in class

Complex numbers

We let $z = x + iy$, $x, y \in \mathbb{R}$. This is the standard form of a complex number. In fact we can think of the complex numbers as pairs of the form (x, y) and we are able to perform the following arithmetic operations.

$$(x,y) + (a,b) = (x+a, y+b)$$

$$\lambda(x,y) = (\lambda x, \lambda y).$$

$$(x,y)(a,b) = (ax - by, xb + ay)$$

Notice $(0,1)(0,1) = (0 \times 0 - 1 \times 1, 0 + 0)$

$$= (-1, 0).$$

Now we identify $(x, 0)$ with $\mathbb{R}$

So $(0,1)(0,1) = (0,1)^2 = (-1,0)$

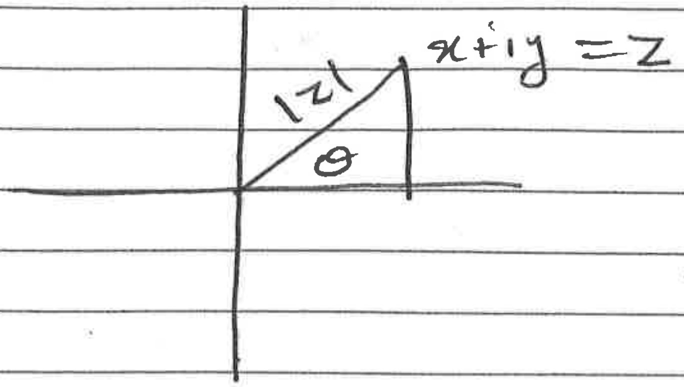
Thus $(0,1)$ is the square root of $(-1,0)$

Identify $(0,1)$ with i and $(-1,0)$ with -1.

Hence $i^2 = -1$.

or $i = \sqrt{-1}$

We also have the polar form of a complex number.



Now the real part of z is x or $Re(z) = x$

The imaginary part of z is y or $Im(z) = y$.

The <u>modulus</u> of z is $|z| = \sqrt{x^2+y^2}$.

On the diagram, θ is the argument of z.
More precisely it is the <u>principal value</u> of the argument.

Clearly $x = |z|\cos\theta$
$y = |z|\sin\theta$.

If $r = |z|$, then

$$z = r(\cos\theta + i\sin\theta)$$

Notice that if $\varphi = \theta + 2\pi$
Then by periodicity

$$z = r(\cos(\theta+2\pi) + i\sin(\theta+2\pi))$$

$$= r(\cos\theta + i\sin\theta).$$

Adding 2π to the argument revolves us back to where we started.

The value of θ lying in the interval $[-\pi, \pi)$ is called the principal value of the argument

$z = r(\cos\theta + i\sin\theta)$ is called the polar form of the complex number.

The set of all arguments is

$$\arg(z) = \{\theta : z = r(\cos\theta + i\sin\theta)\}$$

We can find θ by trigonometry.

For example, if $x, y \in (0, \infty)$

$$x + iy = \sqrt{x^2+y^2}(\cos\theta + i\sin\theta)$$

or $$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}.$$

The solution of these equations with $\theta \in [-\pi, \pi)$ is the principal value of θ.

The most important result involving complex numbers is <u>Euler's Formula</u>.

Euler's Formula

For all θ

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We will give a more precise proof later.

For now, we use the fact that the equation

$$y' = ay, \quad y(0) = y_0$$

has a unique solution.

Let $y(\theta) = e^{i\theta}$

Then $\frac{d}{d\theta} y(\theta) = ie^{i\theta}$
$= iy$.

and $y(0) = 1$.

Now consider $h(\theta) = \cos\theta + i\sin\theta$

Notice $h'(\theta) = -\sin\theta + i\cos\theta$
$= i(i\sin\theta + \cos\theta)$
$= ih$

and $h(0) = \cos 0 + i\sin 0 = 1$

h and y both satisfy the same problem, so they must be equal.

$\therefore$ $e^{i\theta} = \cos\theta + i\sin\theta$.

This is amazingly useful and is a good place to start.