# Complex Analysis



**Definition** A complex number $z$ has the form $z = a + ib$ where $a \in \mathbb{R}$, $b \in \mathbb{R}$, $i = \sqrt{-1}$.

Where does $i$ come from? Cardano applied a formula for the roots of a ~~ch~~ cubic (see notes) to solve

$$x^3 - 15x - 4 = 0$$

$x = 4$ is a solution. He found

$$x = \left(2 + \sqrt{-121}\right)^{1/3} + \left(2 - \sqrt{-121}\right)^{1/3}$$

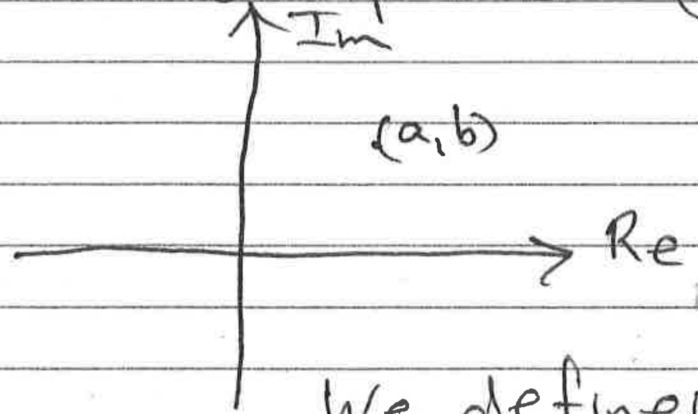
In fact this gives $x = 4$, because

$$(2 \pm i)^3 = 2 \pm 11i$$

$$\therefore \quad (2 \pm 11i)^{1/3} = (2 \pm i) \quad \text{so}$$

$$x = 2 + i + 2 - i = 4$$

After many years, the nature of complex numbers was worked out geometrically. The complex plane is the set of points $(a, b)$




This is an extension of the number line.

We define (i) $(a, b) + (c, d) = (a + c, b + d)$

(ii) $\lambda(a, b) = (\lambda a, \lambda b)$

(iii) $(a, b)(c, d) = (ac - bd, ad + bc)$

Identify $\mathbb{R}$ with the points $(a, 0)$

Now $(0,1)^2 = (0,1)(0,1) = (0 - 1, 0) = (-1, 0)$

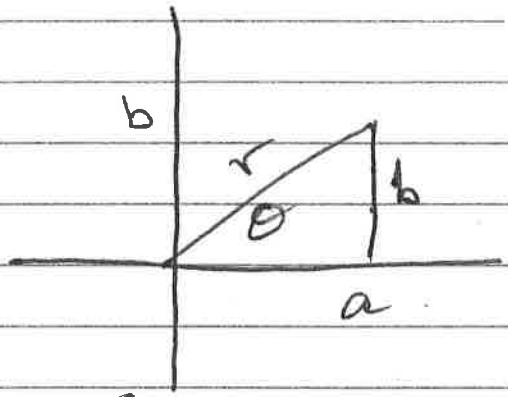
We call $(0,1)$ the square root of $-1$ and denote it by $i$.

Then $(a,b) = (a,0) + i(0,b)$ which we write as $a+ib$.

The real part of $z = a+ib$ is $\mathrm{Re}(z) = a$, the imaginary part is $\mathrm{Im}(z) = b$.

The polar form of a complex number $z$ is

$$z = r(\cos\theta + i\sin\theta)$$




$r = \sqrt{a^2+b^2}$ the <u>modulus</u> of $z$

we write $\sqrt{a^2+b^2} = |z|$.

Clearly $a = r\cos\theta$, $b = r\sin\theta$

So $a+ib = r(\cos\theta + i\sin\theta)$.

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$ in first quadrant
adjust according to quadrant

The complex conjugate of $z$ is
$\bar{z} = a - ib$

The <u>complex</u> <u>number</u> <u>cannot</u> <u>be ordered</u> in terms of the $\gtrless$ signs.

Suppose $i > 0$. Thus
$i.i = -1 > i.0 = 0$

This is nonsense

Suppose $i < 0$. Hence $i i = -1 > i.0 = 0$

∴ A contradiction. This proves the claim.

The value of $\theta$ in "$z = re^{i\theta}$" is called the argument of $z$.
("$re^{i\theta}$" we will come to this shortly)

Now $\cos(\theta + 2k\pi) = \cos\theta$, $k \in \mathbb{Z}$
$\sin(\theta + 2k\pi) = \sin\theta$

So $\theta$ is not unique. We pick an interval of length $2\pi$ for $\theta$ to lie in. Usually

$$\theta \in (-\pi, \pi] \quad (\text{or } \theta \in [-\pi, \pi))$$

We will use → This is called the <u>principal value</u> of $\theta$. We could take $[0, 2\pi)$, some authors do

We denote $\operatorname{Arg} z = \{\theta,\ z = r(\cos\theta + i\sin\theta),\ \theta \in (-\pi, \pi]\}$

$$\arg z = \{\operatorname{Arg} z + 2k\pi,\ k \in \mathbb{Z}\}$$

<u>Euler's Formula</u> $e^{i\theta} = \cos\theta + i\sin\theta$

<u>Proof</u> $y'(\theta) = iy(\theta)$, $y(0) = 1$ has a unique solution.
Clearly if $y(\theta) = e^{i\theta}$, then $y' = iy$

$y(0) = 1$. But if $h(\theta) = \cos\theta + i\sin\theta$

$$h'(\theta) = -\sin\theta + i\cos\theta$$
$$= i(\cos\theta + i\sin\theta) = ih$$

and $h(0) = 1$

So $h$ and $g$ must be the same.

<u>Corollary</u> De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

<u>Proof</u> $(e^{i\theta})^n = e^{ni\theta} = \cos(n\theta) + i\sin(n\theta)$

## Functions of a complex variable

The simplest examples are polynomials. These are nice because they are unique.

eg. $f(z) = z^2$, for each $z$ there is one possible value. However other functions may not have unique values

eg. $f(z) = \sqrt{z}$. This has two possible values. We need to specify which one we want. This is important. Consider the apparent paradox.

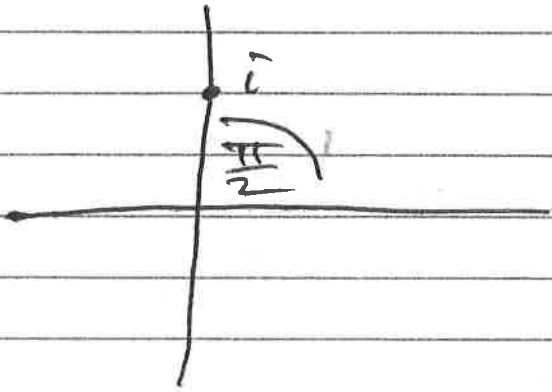
$$-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1.$$

This is wrong because $\sqrt{-1}\sqrt{-1} \neq \sqrt{(-1)(-1)}$. We have to specify which roots we are using. If we write

$$-1 = e^{\pi i} = (\cos\pi + i\sin\pi) = \left(e^{\frac{\pi}{2}i}\right)^2$$
$$= e^{\pi/2 i}\, e^{\pi/2 i}$$
$$= i \times i$$
$$= -1.$$

There is no contradiction here.

$e^{\pi/2 i} = i$




$e^{-\frac{\pi}{2}i} = -i$

$e^{-\pi i} = -1$

Similar statements hold for $n$th roots.

The equation $z^n = 1$ has $n$ solutions $n \geq 1$.

Solving is easy. Take $n=27$
We have $z^{27}=1$.
We know $e^{2k\pi i}=1$, $k \in \mathbb{Z}$

$$z^{27} = e^{2k\pi i}$$
$$z = e^{\frac{2k\pi i}{27}}$$

$k=0$, $z=1$
$k=1$, $z=e^{\frac{2\pi i}{27}}$. Also $z=e^{-\frac{2\pi i}{27}}$ is a root
$k=2$, $z=e^{\frac{4\pi i}{27}}$, Also $z=e^{-\frac{4\pi i}{27}}$ is a root

We can list all 27 roots by letting $k$ take values $1, 2, 3$ ——

Solutions of the equation $z^n=1$ are called the roots of unity. They have nice properties

$$z^n-1=(z-1)(1+z+z^2+ \cdots +z^{n-1})$$

Let $z=\omega \neq 1$, but $\omega^n=1$

So $\omega^n-1=0=(\omega-1)(1+\omega+\omega^2+ \cdots +\omega^{n-1})$
$\omega-1 \neq 0$
So $1+\omega+\omega^2+ \cdots +\omega^{n-1}=0$

eg $n=3$
Take $\omega=e^{\frac{2\pi i}{3}}$
So $1+e^{\frac{2\pi i}{3}}+e^{\frac{4\pi i}{3}}=0$

<u>Logarithms</u> The logarithm of a complex number is defined by

$\log_e(z) = \ln|z| + i\arg(z)$. The principal value is $\ln z = \ln|z| + i\operatorname{Arg}(z)$