Singularities, Poles and Residues

We start with the following integral $f(z) = \frac{1}{z}$, $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$

Then
$$\int_\gamma f(z)\,dz = \int_0^{2\pi} \frac{1}{e^{it}} i e^{it}\,dt$$
$$= i\int_0^{2\pi} dt = 2\pi i$$

Let $f(z) = \frac{1}{z^2}$, γ is the same
$$\int_\gamma f(z)\,dz = \int_0^{2\pi} \frac{1}{(e^{it})^2} i e^{it}\,dt$$
$$= i\int_0^{2\pi} e^{-it}\,dt$$
$$= \left[-e^{-it}\right]_0^{2\pi} = -(e^{-2\pi i} - e^0)$$
$$= 0$$

In fact for $n = 2, 3, 4, \ldots$ $f(z) = \frac{1}{z^n}$
$$\int_\gamma f(z)\,dz = 0$$

Note that $z = 0$ is a singularity for f. In fact, it is a pole.

There are three kinds of singularities that concern us

(1) Removable singularities

(2) Poles

(3) Essential singularities — singularities that are neither (1) nor (2)

Example of (1)
$$f(z) = \frac{\sin z}{z}.$$
$z = 0$ is a singularity.

But $\lim_{z\to 0} \frac{\sin z}{z} = 1$. So we can set

$$f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0. \end{cases}$$

This is continuous, and we have removed the singularity.

Definition Suppose that f is an analytic function with an isolated singularity at z_0. z_0 is a pole of order n (*) if the function $1/f$ is analytic in a region D containing z_0

(*) What is n? We need a result to actually state this.

Theorem Suppose that f is an analytic function on $\Omega \subseteq \mathbb{C}$, with an isolated zero at z_0. Then there is a neighbourhood $D \subset \Omega$ containing z_0 and a nowhere vanishing function g and a unique positive integer n such that

$$f(z) = (z-z_0)^n g(z), \quad \text{all } z \in D$$

Proof Write

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

There is a smallest n such that $a_n \neq 0$

So

$$f(z) = \underbrace{a_0 + \dots +}_{0} a_n (z-z_0)^n + a_{n+1}(z-z_0)^{n+1}$$

$$= (z-z_0)^n (a_n + a_{n+1}(z-z_0) + \dots)$$

$$= (z-z_0)^n g(z)$$

$g(z_0) \neq 0$. So by continuity there is a region around z_0, where g is nonzero

Now suppose that there are two numbers n, m and two functions g, h

$$f(z) = (z-z_0)^n g(z) = (z-z_0)^m h(z) \quad \text{Let } m>n$$

, then

$$g(z) = (z-z_0)^{m-n} h(z)$$

But $m-n>0$. So $g(z_0) = (z_0-z_0)^{m-n} h(z_0) = 0$

A contradiction. Similarly if $n>m$.

So $n=m$. Thus $g=h$ since

$$(z-z_0)^n g(z) = (z-z_0)^n h(z)$$

<u>Theorem</u> Suppose that f has a pole at $z_0 \in \Omega$ Then there is a positive integer n and a neighbourhood $D \subseteq \Omega$ containing z_0 and an analytic function h such that for all $z \in D$

$$f(z) = \frac{h(z)}{(z-z_0)^n}.$$

n is the order of the pole

<u>Proof</u> $1/f$ is analytic at z_0. z_0 is a zero, so we can write

$$1/f(z) = (z-z_0)^n g(z)$$

So

$$f(z) = (z-z_0)^n \frac{1}{g(z)} = \frac{h(z)}{(z-z_0)^n}.$$

A pole of order 1 is a simple pole.

<u>Laurent Series</u> Suppose that f is analytic with a pole of order n at z_0, then there exist numbers $\{a_k\}_{k=-n}^{\infty}$ such that

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}$$
$$+ \sum_{k=0}^{\infty} a_k (z-z_0)^k .$$

This is called a Laurent series.

<u>Proof</u> Let
$$f(z) = (z-z_0)^{-n} h(z)$$
$$= (z-z_0)^{-n} \sum_{k=0}^{\infty} \bar{a}_k (z-z_0)^k$$
$$= (z-z_0)^{-n} \left[\bar{a}_0 + \bar{a}_1 (z-z_0) + \bar{a}_2 (z-z_0)^2 + \cdots \right.$$

Take $\bar{a}_0 = a_{-n}$, $\bar{a}_1 = a_{-n+1}$ etc

<u>Definition</u> The number a_{-1} is called the residue of f at z_0.
$$\text{Res}(f, z_0) = a_{-1}$$

<u>Example</u>
$$f(z) = \frac{e^z}{z^3} = \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)$$
$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{3!} + \frac{z}{4!} + \cdots$$

order of the pole is 3
$$\text{Res}(f, z_0 = 0) = \frac{1}{2}.$$

Example $f(z) = \frac{\cos z}{z}$.

$$= \frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)$$

$$= \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \cdots$$

pole at $z=0$ of order 1, residue is 1.

Example $f(z) = \frac{\cos z}{z^2}$

$$= \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)$$

pole of order 2, residue is zero.

Example $f(z) = \frac{\sin z}{z^4}$

$$= \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{5!} - \cdots$$

order 3, $\mathrm{Res}(f, 0) = -\frac{1}{6}$.

Example $f(z) = \frac{\cos z \sin z}{z^4}$

$$= \frac{\frac{1}{2}\sin(2z)}{z^4}$$

$$= \frac{1}{z^4}\,\frac{1}{2}\left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots\right)$$

$$= \frac{1}{z^3} - \frac{4}{6z} + \cdots$$

order 3, $\mathrm{Res}(f, 0) = -2/3$

We can generalise Laurent series to ones with infinitely many powers of $\frac{1}{z-z_0}$

<u>Example</u> $\frac{1}{1-z} = 1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n$

$|z| < 1$.

However $\frac{1}{(1-z)} = \frac{1}{z(\frac{1}{z}-1)} = -\frac{1}{z(1-1/z)}$

$= -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right) \qquad |z| > 1$

$= -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \dots$

<u>Caution</u> $z = 0$ is <u>not</u> a pole.

<u>Recall</u> that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$|x| < 1$. Since

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt \qquad |x| < 1$$

$$= \int_0^x (1 - t^2 + t^4 - t^6 + \dots)\,dt$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Now

$$\frac{\pi}{2} - \tan^{-1} x = \int_x^{\infty} \frac{1}{t^2(1+\frac{1}{t^2})}\,dt \qquad |x| > 1$$

$$= \int_x^{\infty} \frac{1}{t^2}\left(1 - \frac{1}{t^2} + \frac{1}{t^4} - \frac{1}{t^6} + \dots\right)dt$$

$$\frac{\pi}{2} - \tan^{-1} x = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \dots$$

$$\left[\int_x^{\infty} \frac{dt}{1+t^2} = \tan^{-1} t\Big]_x^{\infty} = \tan^{-1}\infty - \tan^{-1} x = \frac{\pi}{2} - \tan^{-1} x \right]$$

What if we do not have a Laurent Series available?

Theorem Suppose that f has a pole at z_0. If z_0 is a simple pole then

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

If the pole is of order n

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{(n-1)}} \left((z - z_0)^n f(z) \right)$$

This is the residue formula

Proof $$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z) \right)$$

Proof $$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

So $$(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + a_{-n+2}(z-z_0)^2 + \cdots + a_{-1}(z-z_0)^{n-1} + \sum_{k=0}^{\infty} a_k (z-z_0)^{k+n}$$

$$\lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (\;\;) = (n-1)!\, a_{-1} + \lim_{z \to z_0} \sum_{k=0}^{\infty} (k+n) \ldots (k+n-1)\, a_k (z-z_0)^{k+1}$$

$$= (n-1)!\, a_{-1} + 0$$

$$\therefore\ a_{-1} = \frac{1}{(n-1)!} \cdot \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z-z_0)^n f(z) \right)$$

Example

$$f(z) = \frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z+i)(z-i)}$$ poles at $z = \pm i$

Simple poles

$$\text{Res}(f, i) = \lim_{z \to i} (z-i) f(z)$$

$$= \lim_{z \to i} \frac{e^{iz}\cancel{(z-i)}}{(z+i)\cancel{(z-i)}}$$

$$= \lim_{z \to i} \frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i}$$

$\text{Res}(f, -i)$