Example

$$f(z) = \frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z+i)(z-i)}$$ poles at $z = \pm i$

Simple poles

$$\text{Res}(f, i) = \lim_{z\to i} (z-i) f(z)$$

$$= \lim_{z\to i} \frac{e^{iz}\cancel{(z-i)}}{(z+i)\cancel{(z-i)}}$$

$$= \lim_{z\to i} \frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i}$$

$$\text{Res}(f,-i) = \lim_{z\to -i} \frac{e^{iz}\cancel{(z+i)}}{\cancel{(z+i)}(z-i)}$$

$$= \frac{e}{-2i}$$

Now consider

$$f(z) = \frac{e^{iz}}{(z^2+1)^2} = \frac{e^{iz}}{(z+i)^2(z-i)^2}$$ poles at $\pm i$ of order 2

$$\text{Res}(f,i) = \lim_{z\to i} \frac{d}{dz} \frac{\cancel{(z-i)^2} e^{iz}}{(z+i)^2\cancel{(z-i)^2}} = \lim_{z\to i} \frac{ie^{iz}}{(z+i)^2} - \frac{2e^{iz}}{(z+i)^3}$$

$$= \frac{ie^{-1}}{(2i)^2} - \frac{2e^{-1}}{(2i)^3} = \frac{-i}{2e}$$

$$\text{Res}(f,-i) = \lim_{z\to -i} \frac{d}{dz} \frac{(z+i)^2 e^{iz}}{(z+i)^2(z-i)^2} = \lim_{z\to i}\left(\frac{ie^{iz}}{(z-i)^2} - \frac{2e^{iz}}{(z-i)^3}\right)$$

$$= \frac{ie}{(-2i)^2} - \frac{2e}{(-2i)^3} = 0$$

Ex

$$f(z) = \frac{\sin z}{z-1}, \quad \mathrm{Res}(f,1) = \lim_{z\to 1} \frac{(z-1)\sin z}{z-1} = \sin 1$$

To find the order of a pole we can use a Taylor expansion near the pole
Consider $\tan z$, $z = \pi/2$. Now

$$\sin z = 1 - \frac{1}{2}(z-\pi/2)^2 + \frac{1}{4}(z-\pi/2)^4 - \cdots$$

$$\cos z = -(z-\pi/2) + \frac{1}{3!}(z-\pi/2)^3 - \cdots$$

So $\tan z \simeq \dfrac{1-\frac{1}{2}(z-\pi/2)^2}{-(z-\pi/2)}$ z near $\pi/2$ as other terms are small

So $z = \frac{\pi}{2}$ is a simple pole and the residue is -1. We also have

$$\lim_{z\to\frac{\pi}{2}} (z-\pi/2)\frac{\sin z}{\cos z} = -1$$

Now let $P(z) = \dfrac{a_{-n}}{(z-z_0)^n} + \cdots + \dfrac{a_{-1}}{z-z_0}$, let $h(z)$ be analytic. Now let $\gamma(t) = z_0 + Re^{it}$. Then

$$I = \int_\gamma (P(z)+h(z))dz = \int_\gamma P(z)\,dz \quad \text{since}$$

$\int_\gamma h(z)dz = 0$ by Cauchy's Theorem

and $\displaystyle\int_\gamma \frac{1}{(z-z_0)^n}dz = 0 \quad n \geq 2.$

Thus $I = i\displaystyle\int_0^{2\pi} a_{-1}dt = 2\pi i\, a_{-1}$, using the definition of a contour integral.
In fact we have the following

Theorem If f is analytic in Ω except at the poles $z_1, \ldots, z_N$ none of which are on the boundary of Ω. Let γ be the boundary of Ω. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(f, z_k).$$

Example Let $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$

$$f(z) = \frac{e^z}{z^3}.$$

$$\operatorname{Res}(f,0) = \lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2} \frac{z^3 e^z}{z^3} = \frac{1}{2}.$$

$$\therefore \int_\gamma f(z)\,dz = 2\pi i \cdot \frac{1}{2} = \pi i$$