Trigonometric Integrals To evaluate integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$ we make a substitution of the form $z = e^{i\theta}$, $\theta \in [0, 2\pi)$. This turns the real integral into a contour integral around C the unit circle.

$dz = ie^{i\theta}d\theta = iz\,d\theta$. Hence

$$d\theta = \frac{dz}{iz}.$$

$$\frac{1}{z} = e^{-i\theta} \therefore z + \frac{1}{z} = e^{i\theta} + e^{-i\theta}$$
$$= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$
$$= 2\cos\theta$$

$$\therefore \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$z - \frac{1}{z} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$$
$$= 2i\sin\theta.$$

$$\therefore \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

__Example__ Calculate $I = \int_0^{2\pi} (\cos^3 t + \sin^2 t)\,dt$

$$= \int_C \left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)^3 + \left(\frac{1}{2i}\left(z - \frac{1}{z}\right)\right)^2 \frac{dz}{iz}$$

$$= \int_C \left[\frac{1}{8}\left(z^3 + 3z^2 \cdot \frac{1}{z} + 3z \cdot \frac{1}{z^2} + \frac{1}{z^3}\right) - \frac{1}{4}\left(z^2 - 2z \cdot \frac{1}{z} + \frac{1}{z^2}\right)\right] \frac{dz}{iz}$$

Pole at $z = 0$, the $\frac{1}{z}$ term is $\frac{1}{2iz}$.

Residue is $\frac{1}{2i}$

$$\therefore I = 2\pi i \frac{1}{2i} = \pi$$

__Example__ Calculate $I = \int_0^{2\pi} (\cos^4 t + \sin^4 t)\,dt$

$$= \int_0^{2\pi} \left[\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)^4 + \left(\frac{1}{2i}\left(z - \frac{1}{z}\right)\right)^4\right] \frac{dz}{iz}$$

Recall $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n$

So $\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^4 = \frac{1}{2^4}\left(z^4 + 4z^3\cdot\frac{1}{z} + 6\frac{z^2}{z^2} + 4z\cdot\frac{1}{z^3} + \frac{1}{z^4}\right)$

$\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^4 = \frac{1}{2^4}\left(z^4 - 4z^3\cdot\frac{1}{z} + 6z^2\cdot\frac{1}{z^2} - 4\frac{z}{z^3} + \frac{1}{z^4}\right)$

Hence

$$\left(\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^4 + \left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^4\right)\frac{dz}{iz} = \frac{1}{2^4}\left[\left(\frac{z^3}{i} + \cdots + \frac{6}{iz} + \cdots\right) + \left(\frac{z^3}{i} + \cdots + \frac{6}{iz} + \cdots\right)\right]$$

pole is at $z=0$. Residue $= \frac{1}{2^4}\;\frac{12}{i} = \frac{12}{16i}$

$\therefore\; I = 2\pi i\left(\frac{12}{16i}\right) = \frac{3\pi}{2}$.

Exercise $\int_0^{2\pi}(\cos^6 t + \sin^6 t)\,dt$

Example Calculate $I = \int_0^{2\pi}\frac{dt}{a+b\cos t}$, $a>|b|$.

$$\frac{1}{a+b\cos t} = \frac{1}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)} = \frac{z}{az+\frac{bz^2}{2}+\frac{b}{2}}$$

$$= \frac{\frac{2}{b}z}{z^2+\frac{2az}{b}+1}.$$

So $\frac{dt}{a+b\cos t} = \frac{2/b\, z}{z^2+\frac{2az}{b}+1}\;\frac{dz}{iz} = \frac{2}{ib}\;\frac{1}{z^2+\frac{2az}{b}+1}$

$$\int_0^{2\pi}\frac{dt}{a+b\cos t} = \int_C \frac{2}{ib}\;\frac{dz}{z^2+\frac{2a}{b}z+1}$$

We solve $z^2+\frac{2a}{b}z+1 = 0$

$$z = \frac{-a\pm\sqrt{a^2-b^2}}{b},\quad z_1 = \frac{-a-\sqrt{a^2-b^2}}{b}$$

$$z_2 = \frac{-a+\sqrt{a^2-b^2}}{b}, \quad |z_1| = \frac{|a+\sqrt{a^2-b^2}|}{|b|} > \frac{|a|}{|b|} > 1$$

z_1 is outside the unit circle.
z_2 is inside the cirle. (see p111 of notes)

z_2 is a simple pole:

$$f(z) = \frac{1}{z^2 + \frac{2a}{b}z + 1} = \frac{1}{(z-z_1)(z-z_2)}$$

$$\text{Res}(f(z), z_2) = \lim_{z \to z_2} \frac{\cancel{(z-z_2)}}{(z-z_1)\cancel{(z-z_2)}} = \frac{1}{z_2 - z_1}$$

$$= \frac{1}{\left(\frac{-a}{b} + \frac{\sqrt{a^2-b^2}}{b}\right) - \left(-\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b}\right)} = \frac{b}{2\sqrt{a^2-b^2}}$$

So $$I = \frac{2}{ib}\int_c f(z)\,dz = \frac{2}{ib} 2\pi i \frac{b}{2\sqrt{a^2-b^2}}$$
$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example $$I = \int_0^{\pi} \frac{dt}{1+b\cos^2 t} \qquad b \geq 1$$

Notice $\cos^2 t = \frac{1}{2}(1+\cos(2t))$

Put $z = e^{2it}$, $t \in [0, \pi]$

$dz = 2iz\,dt \quad \therefore dt = \dfrac{dz}{2iz}$.

$\frac{1}{2}\left(z + \frac{1}{z}\right) = \cos(2t)$. So $\dfrac{1}{1+b\cos^2 t} =$

$$\frac{1}{1 + b\left(\frac{1}{2}\left(1 + \frac{1}{2}\left(z + \frac{1}{z}\right)\right)\right)}$$

So $$I = \frac{2}{ib}\int_c \frac{1}{(z^2 + (4/b+2)z + 1}\,dz$$

$$\left[\, I = \int_c \frac{1}{1 + \frac{b}{2} + \frac{b}{4}\left(z + \frac{1}{z}\right)} \frac{dz}{2iz} \,\right]$$

Roots of $z^2 + (2+4/b)z + 1 = 0$ are

$$z_1 = -\frac{2-b-2\sqrt{1+b}}{b}, \quad z_2 = -\frac{2-b+\sqrt{1+b}}{b}$$

Clearly $|z_1| > 1$, $|z_2| < 1$ (notes p113)

$$f(z) = \frac{1}{z^2 + (2+4/b)z + 1}$$

$$\operatorname{Res}(f(z), z_2) = \lim_{z \to z_2} \frac{\cancel{(z-z_2)}}{(z-z_1)\cancel{(z-z_2)}}$$

$$= \frac{1}{z_2 - z_1}$$

$$= \quad 1 \quad = \frac{b}{4\sqrt{1+b}}.$$

Thus

$$I = \frac{2}{ib} \int_C f(z)\,dz$$

$$= \frac{2}{ib}\, 2\pi i \,\frac{b}{4\sqrt{1+b}} = \frac{\pi}{\sqrt{1+b}}.$$

Integration on the real line. It is possible to evaluate many integrals on $\mathbb{R}$ using residues.

Example Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Recall $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{0} f(x)\,dx + \lim_{T\to\infty} \int_0^T f(x)\,dx$$

If this exists, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$