we calculate $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$.

we know $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$, $\lim_{x\to\pm\infty}\tan^{-1}x = \pm\frac{\pi}{2}$

So $\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \lim_{R\to\infty}\tan^{-1}x\Big|_{-R}^{R} = \pi$

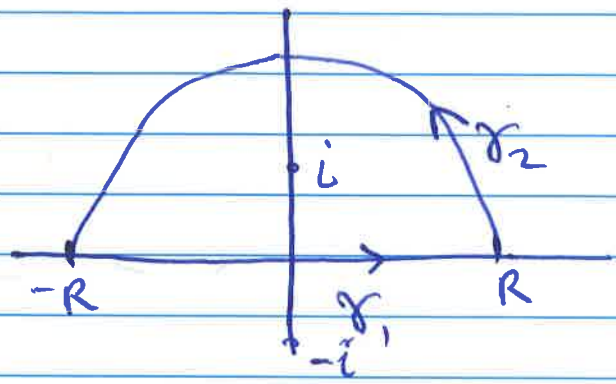
as integral is convergent.

However we can also use complex variables.

$f(z) = \frac{1}{1+z^2}$ has simple poles at $\pm i$

Let $\gamma = \gamma_1 + \gamma_2$, $\gamma_1(t) = t$, $t \in [R, R]$

$\gamma_2(t) = Re^{it}$ $0 \le t \le \pi$



$\int_\gamma f = \int_{-R}^{R}\frac{dt}{1+t^2} + \int_0^{\pi}\frac{iRe^{it}}{R^2e^{2it}+1}dt$

$\mathrm{Res}(f, i) = \lim_{z\to i}\frac{(z-i)}{(z-i)(z+i)} = \frac{1}{2i}$

Now $\left|\int_0^{\pi}\frac{iRe^{it}}{R^2e^{2it}+1}dt\right| \le \int_0^{\pi}\frac{R}{|R^2e^{2it}+1|}dt \le \frac{\pi R}{R^2-1} \to 0$

since $|R^2e^{2it}+1| \ge |R^2-1|$.

Thus letting $R\to\infty$ $\int_{-\infty}^{\infty}\frac{dt}{1+t^2} = 2\pi i\cdot\frac{1}{2i} = \pi$

This method works for a large range of problems. Here is another example. Find $\int_{-\infty}^{\infty}\frac{dx}{x^4+1}$. Use the same γ as before

Simple poles at roots of $z^4+1=0$

$z^4 = e^{\pi i + 2k\pi i}$ roots are $z_1 = e^{\frac{\pi i}{4}}$, $z_2 = e^{-\frac{\pi i}{4}}$
$z_3 = e^{3\pi i/4}$, $z_4 = e^{-3\pi i/4}$

z_1 and z_3 are above axis, inside γ.

$\text{Res}(f(z), z_1) = \lim_{z \to z_1} \frac{z - z_1}{z^4+1} = \lim_{z \to z_1} \frac{1}{4z^3}$ by L'Hôpital's rule $= -\frac{1}{4}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$ A

$\text{Res}(f(z), z_3) = \lim_{z \to z_3} \frac{z - z_3}{z^4+1} = \lim_{z \to z_3} \frac{1}{4z^3}$
$= \frac{1}{4}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$ B

$A + B = -\frac{i}{2\sqrt{2}}$. $\int_\gamma f = \int_{-R}^{R} \frac{dt}{1+t^4} + \int_0^\pi \frac{iRe^{it}}{R^4 e^{4it}+1}\, dt$

So $\int_\gamma f = 2\pi i\left(\frac{-i}{2\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$ by Cauchy residue Theorem.

Now $\left|\int_{\gamma_2} f\right| = \left|\int_0^\pi \frac{iRe^{it}}{R^4 e^{4it}+1}\, dt\right|$

$\leq \frac{\pi R}{R^4 - 1} \to 0$ as $R \to \infty$

Thus taking $R \to \infty$ $\int_\gamma f = \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$.