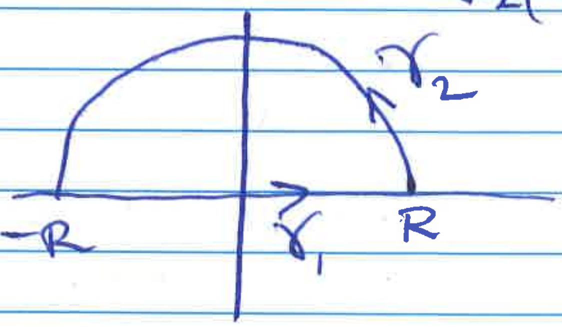
Proposition Suppose that f is analytic in the upper half plane $\text{Im}(z) \geq 0$, except at finitely many poles, none of which lie on the real axis. Suppose further that there is a constant $A > 0$, such that for large enough $R > 0$ $|f(z)| \leq A/R^k$, $k \geq 2$, on the semicircle $z = Re^{it}$, $0 \leq t \leq \pi$.

Then
$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \Sigma$$

where Σ is the sum of the residues in the upper half plane

Proof Let $\gamma_1(t) = t$, $0 \leq t \leq \pi$

$$\gamma_2(t) = Re^{it}$$



$$\gamma = \gamma_1 + \gamma_2$$

$$\int_\gamma f(z)\,dz = \int_{-R}^{R} f(t)\,dt + \int_0^{\pi} f(Re^{it})\, iRe^{it}\,dt$$
$$= 2\pi i \Sigma$$

$$\left|\int_0^{\pi} f(Re^{it})\, iRe^{it}\,dt\right| \leq R\int_0^{\pi} |f(Re^{it})|\,dt$$
$$\leq R\frac{A}{R^k} = \frac{A}{R^{k-1}} \to 0$$
as $R \to \infty$

$$\lim_{R\to\infty} \int_\gamma f = \lim_{R\to\infty} \int_{-R}^{R} f(t)\,dt + \lim_{R\to\infty} \int_0^{\pi} f(Re^{it})\, iRe^{it}\,dt$$
$$= \int_{-\infty}^{\infty} f(t)\,dt = 2\pi i \Sigma$$

<u>Example</u> Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$
$a, b > 0$

Simplepoles at $\pm ia$, $\pm ib$.

$f(x)$ decays as $\frac{1}{x^4}$ So proposition holds

$$\text{Res}(f(z), ia) = \lim_{z \to ia} \frac{\cancel{(z-ia)}}{(z+ia)\cancel{(z-ia)}(z^2+b^2)}$$

$$= \frac{1}{2ia(b^2-a^2)}$$

$$\text{Res}(f(z), ib) = \lim_{z \to ib} \frac{\cancel{(z-ib)}}{(z^2+a^2)\cancel{(z-ib)}(z+ib)}$$

$$= \frac{1}{2ib(a^2-b^2)}.$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \left[\frac{1}{2ia(b^2-a^2)} + \frac{1}{2ib(a^2-b^2)} \right]$$

$$= \pi \left[\frac{1}{a(b-a)(b+a)} - \frac{1}{(b+a)(b-a)b} \right]$$

$$= \frac{\pi}{(b-a)(b+a)} \left[\frac{1}{a} - \frac{1}{b} \right]$$

$$= \frac{\pi}{\cancel{(b-a)}(b+a)} \left(\frac{\cancel{b-a}}{a\,b} \right) = \frac{\pi}{ab(b+a)}$$

<u>Example</u> Calculate

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} \qquad \left(\text{Ans } \frac{7\pi}{50} \right)$$

$$x^2+2x+2 = x^2+2x+1+1 = (x+1)^2+1 = 0$$

$$(x+1)^2 = -1 \quad \therefore \quad x+1 = \pm i.$$

$$x = -1+i, \; -1-i.$$

Also $x = \pm i$. These are poles of order 2

$$\text{Res}(f(z), -1+i) = \lim_{z \to -1+i} \frac{\cancel{(z-(-1+i))}\, z^2}{(z^2+1)^2 \cancel{(z-(-1+i))}(z-(-1-i))}$$

$$= \frac{3}{25} - \frac{4}{25} i \qquad \frac{(-1+i)^2}{((-1+i)^2+1)^2(2i)}$$

$$\text{Res}(f(z), i) = \lim_{z \to i} \frac{d}{dz} \frac{\cancel{(z-i)^2}\, z^2}{\cancel{(z-i)^2}(z+i)^2(z^2+2z+2)}$$

$$= \lim_{z \to i} \frac{2z(z^3+z^2-iz-2i)}{(z+i)^3(z^2+2z+2)^2}$$

$$= -\frac{3}{25} + \frac{9}{100} i$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \left(\cancel{\frac{3}{25}} - \frac{4}{25} i - \cancel{\frac{3}{25}} + \frac{9}{100} i \right)$$

$$= \frac{7\pi}{50}$$

Example $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$

$$f(z) = \frac{1}{z^6+1}. \qquad \int_0^\infty \frac{dx}{x^6+1} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^6+1}$$

since f is even

Poles are at roots of $z^6+1=0$

Let $z^6 = e^{\pi i + 2k\pi i}$

$k=0$, $z_1 = e^{\pi i/6}$, $z_2 = e^{\frac{-\pi i}{6}}$

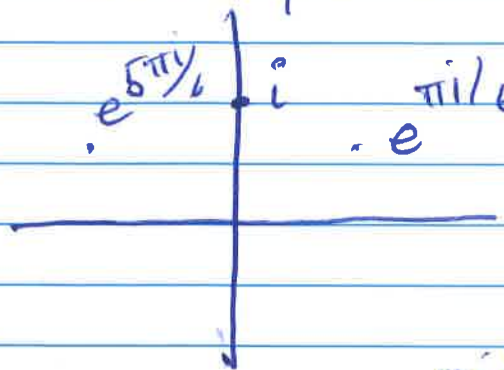
$k=1, \quad z^6 = e^{3\pi i}, \quad z_3 = e^{\frac{\pi i}{2}} = i$

$z_4 = -i,$

$k=2 \quad z^6 = e^{5\pi i}, \quad z_5 = e^{\frac{5\pi i}{6}}, \; z_6 = e^{-\frac{5\pi i}{6}}.$

$z^6+1 = (z-z_1)(z-z_2) \ldots (z-z_6)$

So poles are simple.



z_1, z_3, z_5 are above the axis

$$\mathrm{Res}(f(z), e^{\pi i/6}) = \lim_{z \to e^{\pi i/6}} \frac{z - e^{\pi i/6}}{z^6+1}$$

(By L'Hôpital)

$$= \lim_{z \to e^{\pi i/6}} \frac{1}{6z^5}$$

$$= \frac{1}{6e^{5\pi i/6}} = \frac{1}{6} e^{-\frac{5\pi i}{6}}.$$

$$\mathrm{Res}(f(z), i) = \lim_{z \to i} \frac{z-i}{z^6+1} = \lim_{z \to i} \frac{1}{6z^5}$$

$$= \frac{1}{6i}$$

$$\mathrm{Res}(f(z), e^{\frac{5\pi i}{6}}) = \lim_{z \to e^{5\pi i/6}} \frac{z - e^{5\pi i/6}}{z^6+1}$$

$$= \lim_{z \to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-\frac{\pi i}{6}}.$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\pi i\left(\frac{1}{6}e^{-\frac{5\pi i}{6}} + \frac{1}{6i} + \frac{1}{6}e^{-\frac{\pi i}{6}}\right)$$

$$= 2\pi i\left(-\frac{\sqrt{3}}{12} - \frac{i}{12} - \frac{1}{6}i + \frac{\sqrt{3}}{12} - \frac{i}{12}\right)$$

$$= 2\pi i\left(-\frac{1}{3}i\right) = \frac{2\pi}{3}$$

$$\therefore \int_0^{\infty} \frac{dx}{x^6+1} = \frac{1}{2} \times \frac{2\pi}{3} = \frac{\pi}{3}.$$

Exercise $\displaystyle\int_0^{\infty} \frac{dx}{x^8+1} = \frac{\pi}{8\sin\left(\frac{\pi}{8}\right)}$

Integrals of the form $\displaystyle\int_{-\infty}^{\infty} f(x)e^{+iyx}\,dx$

These are Fourier transforms. Find

$\displaystyle\int_{-\infty}^{\infty} \frac{e^{iyx}}{(x^2+a^2)(x^2+b^2)}\,dx$, $y>0$ poles are at $\pm ia$, $\pm ib$

$$\operatorname{Res}\left(\frac{e^{iyx}}{(x^2+a^2)(x^2+b^2)}, ia\right) = \lim_{x\to ia} \frac{\cancel{(x-ia)}e^{iyx}}{\cancel{(x-ia)}(x+ia)(x^2+b^2)}$$

$$= \frac{e^{-ay}}{2ia(b^2-a^2)}$$

$$\operatorname{Res}(\quad, ib) = \frac{e^{-by}}{2ib(a^2-b^2)}$$

So $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iyx}}{(x^2+a^2)(x^2+b^2)}\,dx = 2\pi i \sum \text{res}$

$$= \frac{\pi}{b^2-a^2}\left(\frac{e^{-ay}}{a} - \frac{e^{-by}}{b}\right)$$

<u>Proposition</u> Suppose that f is analytic in the upper half plane $\mathrm{Im}(z) \geq 0$ except at finitely many poles, none of which lie on the real axis. Suppose further that there is a constant $A > 0$, such that for large enough $R > 0$, $|f(z)| \leq \frac{A}{|z|}$ for z on the semicircle $z = Re^{it}$, $t \in [0, \pi]$

Then $\int_{-\infty}^{\infty} f(x) e^{iyx} dx = 2\pi i \sum$

<u>Example</u> Calculate $\int_{-\infty}^{\infty} \frac{x^3 \cos x}{(x^2+a^2)(x^2+b^2)} dx$

and $\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+a^2)(x^2+b^2)} dx$

Let $f(z) = \frac{z^3 e^{iz}}{(z^2+a^2)(z^2+b^2)}$. We want poles at $+ia$, $+ib$

$$\mathrm{Res}(f(z), ia) = \lim_{z \to ia} \frac{(z-ia) z^3 e^{iz}}{(z-ia)(z+ia)(z^2+b^2)} = \frac{a^2 e^{-a}}{2(a^2-b^2)}$$

$$\mathrm{Res}(f(z), ib) = \lim_{z \to ib} \frac{(z-ib) z^3 e^{iz}}{(z^2+a^2)(z-ib)(z+ib)} = \frac{b^2 e^{-b}}{2(b^2-a^2)}$$

$$\therefore \int_{-\infty}^{\infty} f(x) dx = \pi i \left(\frac{a^2 e^{-a} - b^2 e^{-b}}{a^2 - b^2} \right)$$

$$\int_{-\infty}^{\infty} - \cos x \, dx = 0, \quad \int_{-\infty}^{\infty} - \sin x \, dx = \pi \left(\quad \right)$$