Example Show that $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\,dx = \frac{\pi}{\sin(\pi a)}$

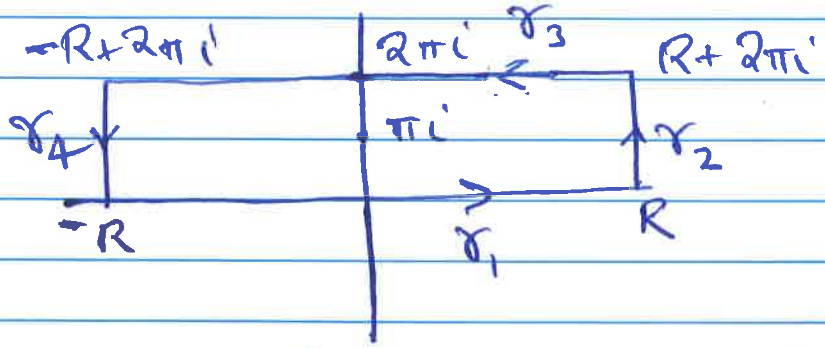
for $0 < a < 1$, $f(z) = \dfrac{e^{az}}{e^z+1}$. Pole of order 1 at $z = \pi i$

$$\operatorname{Res}(f(z), \pi i) = \lim_{z\to\pi i} \frac{(z-\pi i)e^{az}}{e^z+1}$$

$$= \lim_{z\to\pi i} \frac{e^{az} + a(z-\pi i)e^{az}}{e^z}$$

$$= \frac{e^{ia\pi}}{e^{\pi i}} = -e^{ia\pi}$$

We let $\gamma = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$ $\quad (e^{2\pi i} = 1)$



$\gamma_1(x) = x, \ -R \le x \le R$

$\gamma_2(x) = R + ix, \ 0 \le x \le 2\pi$

$\gamma_3(x) = x + 2\pi i, \ -R \le x \le R$

$\gamma_4(x) = -R + ix, \ 0 \le x \le 2\pi$

$$\int_\gamma f = \left(\int_{\gamma_1} + \int_{\gamma_2} - \int_{\gamma_3} - \int_{\gamma_4}\right) f = -2\pi i\, e^{ia\pi}$$

$$\int_{\gamma_1} f = \int_{-R}^{R} \frac{e^{ax}}{e^x+1}\,dx$$

$$\int_{\gamma_3} f = \int_{-R}^{R} \frac{e^{ax+2\pi i a}}{e^{x+2\pi i}+1}\,dx = e^{2\pi i a}\int_{-R}^{R}\frac{e^{ax}}{e^x+1}\,dx$$

So $\displaystyle\int_{\gamma_1} - \int_{\gamma_3} = \left(1 - e^{2\pi i a}\right)\int_{-R}^{R} \frac{e^{ax}}{e^x+1}\,dx$

$$\int_{\gamma_2} f = i\int_0^{2\pi} \frac{e^{aR+iax}}{e^{R+ix}+1}\,dx$$

$$\left|\int_{\gamma_2} f\right| \le \int_0^{2\pi} \frac{e^{aR}}{|e^{R+ix}+1|}\,dx$$
$$\le \int_0^{2\pi} \frac{e^{aR}}{|e^R-1|}\,dx = \frac{2\pi e^{aR}}{e^R-1},$$

So $\frac{e^{aR}}{e^R} = e^{(a-1)R} = e^{-\alpha R}$, $\to 0$ as $R\to\infty$, since $0<a<1$, $\alpha>0$ and $e^{-\alpha R}\to 0$

So Thus $\int_{\gamma_2} f \to 0$ as $R\to\infty$

Similarly $\int_{\gamma_4} f \to 0$ as $R\to\infty$

$$\lim_{R\to\infty}\int_\gamma f = (1-e^{2\pi i a})\int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}\,dx = -2\pi i \times e^{\pi i a}$$

Hence
$$\int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}\,dx = \frac{-2\pi i\, e^{\pi i a}}{1-e^{2\pi i a}}$$
$$= \frac{-2\pi i\,\cancel{e^{\pi i a}}}{\cancel{e^{\pi i a}}(e^{-\pi i a}-e^{\pi i a})}$$
$$= \frac{2\pi i}{e^{\pi i a}-e^{-\pi i a}}$$

$$\sin(\pi a) = \frac{e^{\pi i a}-e^{-\pi i a}}{2i}$$

So
$$\int_{-\infty}^{\infty}\frac{e^{ax}}{e^x+1}\,dx = \frac{\pi}{\sin(\pi a)}.$$

<u>Example</u> Show that $\displaystyle\int_{-\infty}^{\infty}\frac{e^{-2\pi i x\xi}}{\cosh(\pi x)}\,dx = \frac{1}{\cosh(\pi\xi)}$

Poles at $i/2$, $3i/2$. Simple poles

$$f(z) = \frac{e^{-2\pi i z}}{\cosh(\pi z)}. \quad \mathrm{Res}(f, i/2) = \frac{1}{\pi i}e^{\pi\xi}$$
$$\mathrm{Res}(f, 3i/2) = \frac{i}{\pi}e^{3\pi\xi}$$

This is an exercise $\gamma = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$

$\gamma_1(x) = x, \quad -R \le x \le R, \quad \gamma_2(x) = R + ix \quad 0 \le x \le 2$

$\gamma_3(x) = x + 2i, \quad -R \le x \le R, \quad \gamma_4(x) = -R + ix, \quad 0 \le x \le 2$

$$\int_\gamma f = 2\pi i \left(\frac{1}{i\pi} e^{\pi\xi} + \frac{i}{\pi} e^{3\pi\xi} \right)$$

$$=$$

$$= -2e^{\pi\xi}\left(e^{2\pi\xi} - 1\right)$$

$$\int_{\gamma_1} f = \int_{-R}^{R} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)}\, dx$$

$$\int_{\gamma_3} f = \int_{-R}^{R} \frac{e^{-2\pi i(x+2i)\xi}}{\cosh(\pi(x+2i))}\, dx$$

$$\cosh(\pi(x+2i)) = \frac{1}{2}\left(e^{\pi x + 2\pi i} + e^{-\pi x - 2\pi i}\right)$$

$$= \cosh(\pi x)$$

So $$\int_{\gamma_1} - \int_{\gamma_3} = e^{4\pi\xi} \int_{-R}^{R} \frac{e^{-2\pi i x}}{\cosh(\pi x)}\, dx$$

$$\int_{\gamma_2} f = \int_0^2 \frac{e^{-2\pi i(R+ix)\xi}}{\cosh(\pi(R+ix))}\, dx$$

$$|\cosh(\pi(R+ix))|^2 = \cosh^2(\pi R)\cos^2(\pi x) + \sinh^2(\pi R)\sin^2(\pi x)$$
$$= \cosh^2(\pi R)\cos^2(\pi x) + \sinh^2(\pi R)(1 - \cos^2(\pi x))$$
$$= \cos^2(\pi x) + \sinh^2(\pi R)$$
$$\ge \sinh^2(\pi R)$$

$$\therefore \quad |\cosh(\pi(R+ix))| \ge |\sinh(\pi R)|$$

$$\left|\int_{\gamma_2} f\right| = \left|\int_0^2 \frac{e^{-2\pi i(R+ix)\xi}}{\cosh(\pi(R+ix))}dx\right|$$

$$\leq \int_0^2 \frac{e^{4\pi\xi}}{\sinh(\pi R)}dx$$

$$= \frac{2e^{4\pi\xi}}{\sinh(\pi R)} \to 0$$

as $R \to \infty$. Similarly $\int_{\gamma_4} f \to 0$ as $R \to \infty$

Thus $$\left(1 - e^{4\pi\xi}\right)\int_{-\infty}^{\infty} \frac{e^{-2\pi i x}}{\cosh(\pi x)}dx = -2e^{\pi\xi}(e^{2\pi\xi}-1)$$

or $$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x}}{\cosh(\pi x)}dx = \frac{-2e^{\pi\xi}(e^{2\pi\xi}-1)}{1-e^{4\pi\xi}}$$

$$= \frac{-2e^{2\pi\xi}(e^{\pi\xi}-e^{-\pi\xi})}{1-e^{4\pi\xi}}$$

$$= \frac{2\cancel{e^{2\pi\xi}}(e^{\pi\xi}-e^{-\pi\xi})}{\cancel{e^{2\pi\xi}}(e^{2\pi\xi}-e^{-2\pi\xi})}$$

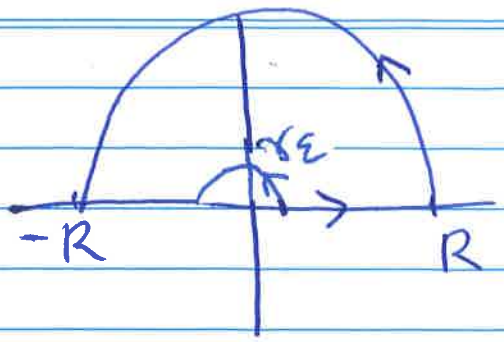
$$= \frac{2(e^{\pi\xi}-e^{-\pi\xi})}{(e^{\pi\xi}+e^{-\pi\xi})(e^{\pi\xi}-e^{-\pi\xi})} = \frac{1}{\cosh(\pi\xi)}$$

<u>Indented contours</u> Evaluate the Dirichlet integral $\int_{-\infty}^{\infty} \frac{\sin x}{x}dx$

Set $$f(z) = \frac{e^{iz}}{z} = \frac{1}{z}\left(1 + iz - \frac{z^2}{2!} + \cdots\right)$$

$$= \frac{1}{z} + k(z)$$ Pole at $z=0$. Residue is 1

But pole is on the real axis. We indent the contour



$\gamma_\varepsilon(t) = \varepsilon e^{it}, \quad t \in [0,\pi]$

$\gamma_1(t) = t \quad -R \le t \le -\varepsilon$

$\gamma_2(t) = t \quad \varepsilon \le t \le R$

$\gamma_3(t) = Re^{it} \qquad t \in [0,\pi)$

$$\gamma = \gamma_1 - \gamma_\varepsilon + \gamma_2 + \gamma_3$$

$$\int_\gamma f = \int_{\gamma_1} - \int_{\gamma_\varepsilon} + \int_{\gamma_2} + \int_{\gamma_3} = 0$$

$$\int_{\gamma_3} f \to 0 \quad \text{as } R \to \infty$$

$$\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} = \int_{\gamma_\varepsilon}$$

So
$$\int_{\gamma_\varepsilon} = \int_0^\pi \frac{e^{i\varepsilon e^{it}}}{\varepsilon e^{it}} i\varepsilon e^{it}\, dt$$

$$= i\int_0^\pi e^{i\varepsilon e^{it}}\, dt$$

$$\to i\int_0^\pi dt = i\pi \quad \text{as } \varepsilon \to 0$$

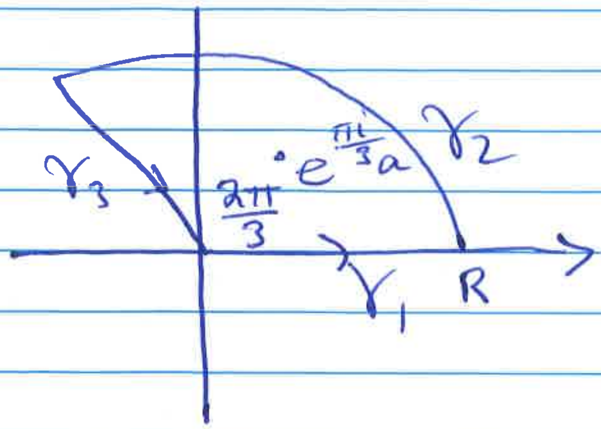
Take $\lim\limits_{R\to\infty} \lim\limits_{\varepsilon\to 0} \int_\gamma = i\pi$

$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x}\, dx = i\pi$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$$

However $PV\int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx = 0$, but $\int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx$ does not exist.

Calculate $\int_0^\infty \frac{dx}{x^3+a^3}$, $a>0$.



$\left[\left(e^{\frac{2\pi i}{3}}\right)^3 = 1\right]$

$\gamma_1(t) = t$, $0 \le t \le R$

$\gamma_2(t) = Re^{it}$, $0 \le t \le \frac{2\pi}{3}$

$\gamma_3(t) = e^{\frac{2\pi i}{3}}t$, $0 \le t \le R$.

$\gamma = \gamma_1 + \gamma_2 - \gamma_3$ $\qquad f(z) = \frac{1}{z^3+a^3}$

Poles at $z_1 = -a$, $z_2 = ae^{\pi i/3}$, $z_3 = ae^{\frac{-\pi i}{3}}$

$$\operatorname{Res}(f, z_2) = \lim_{z\to z_2} \frac{(z-z_2)}{z^3+a^3} = \lim_{z\to z_2} \frac{1}{3z^2}$$

$$= \frac{1}{3a^2 e^{2\pi i/3}} = \frac{e^{-2\pi i/3}}{3a^2}$$

Thus $\int_\gamma f = \frac{2\pi i\, e^{-2\pi i/3}}{3a^2}$

$$\int_{\gamma_1} = \int_0^R \frac{1}{t^3+a^3}\,dt, \quad \left|\int_{\gamma_2}\right| = \left|\int_0^{\frac{2\pi}{3}} \frac{iRe^{it}}{R^3e^{3it}+a^3}\,dt\right|$$

$$\le \int_0^{\frac{2\pi}{3}} \frac{R}{|R^3e^{3it}+a^3|}\,dt = \int_0^{\frac{2\pi}{3}} \frac{R}{|R^3-a^3|}\,dt$$

$$= \frac{2\pi}{3}\frac{R}{|R^3-a^3|} \to 0 \text{ as } R\to\infty$$

$$\int_{\gamma_3} = \int_0^R \frac{e^{\frac{2\pi i}{3}}}{(te^{2\pi i/3})^3+1}\,dt = e^{\frac{2\pi i}{3}}\int_0^R \frac{dt}{t^3+1}$$

So $$\int_\gamma f = \left(1-e^{\frac{2\pi i}{3}}\right)\int_0^R \frac{dt}{t^3+1} + \int_{\gamma_2} f$$

$$\to \left(1-e^{2\pi i/3}\right)\int_0^\infty \frac{dt}{t^3+1} = \frac{2\pi i\, e^{-\frac{2\pi i}{3}}}{3a^2}$$

$$\therefore \int_0^\infty \frac{dt}{t^3+1} = \frac{2\pi i\, e^{-\frac{2\pi i}{3}}}{3a^2\left(1-e^{2\pi i/3}\right)}$$

$e^{-\pi i} = -1$

$$= \frac{2\pi i\, e^{-\frac{2\pi i}{3}}}{3a^2 e^{\pi i/3}\left(e^{-\pi i/3}-e^{\pi i/3}\right)}$$

$$= \frac{2\pi i}{3a^2\left(e^{\pi i/3}-e^{-\pi i/3}\right)}$$

$$= \frac{\pi}{3a^2 \sin(\pi/3)} = \frac{2\pi}{3\sqrt{3}\,a^2}.$$

Finally, Show that

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx = \pi \ln 2$$

$$f(z) = \frac{\ln(z+i)}{z^2+1}.$$ Because $z=i$ is a pole

Let $\gamma = \gamma_1 + \gamma_2$ $\quad \gamma_1(x) = x,\ -R \le x \le R$

$\gamma_2(x) = Re^{ix},\ 0 \le x \le \pi.$

$$\text{Res}(f,i) = \lim_{z\to i} \frac{\cancel{(z-i)}\ln(z+i)}{\cancel{(z-i)}(z+i)}$$

$$= \frac{\ln(2i)}{2i}.$$

In notes we show $\int_{\gamma_2} f \to 0$ as $R\to\infty$

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Now $$\int_{-R}^{R}\frac{\ln(x+i)}{x^2+1}\,dx = \int_{-R}^{0}\frac{\ln(x+i)}{x^2+1}\,dx + \int_{0}^{R}\frac{\ln(x+i)}{x^2+1}\,dx$$

$x \to -x$ in first int.

$$= \int_0^R \frac{\ln(i-x)}{x^2+1}\,dx + \int_0^R \frac{\ln(x+i)}{x^2+1}\,dx$$

$$= \int_0^R \frac{\ln(i-x)+\ln(x+i)}{x^2+1}\,dx$$

$$= \int_0^R \frac{\ln(x-i)+\ln(x+i)+C}{x^2+1}\,dx$$

$\left(\ln(i-x) = \ln(x-i)+\ln(-1)\right)$
$C = \ln(-1) = \pi i$

$$= \int_0^R \frac{\ln(x^2+1)+i\pi}{x^2+1}\,dx$$

$$= 2\pi i\,\frac{\ln(2i)}{2i}$$

$$= \frac{2\pi i}{2i}\left(\ln 2 + \frac{\pi i}{2}\right)$$

$$= \pi\ln 2 + \frac{1}{2}\pi^2 i$$

Let $R \to \infty$ take real part

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx = \pi\ln 2$$

and $$\int_0^\infty \frac{i\pi}{x^2+1}\,dx = \frac{i}{2}\pi^2$$