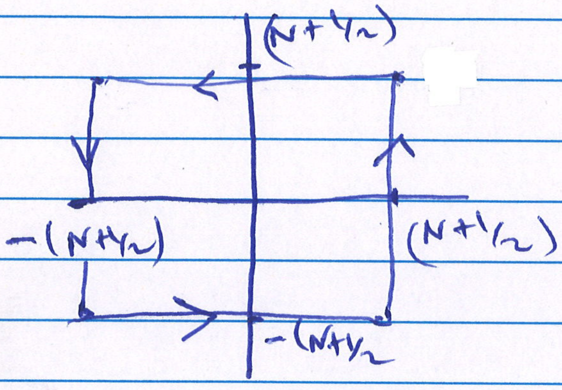
Summation of Series It is a common problem to sum infinite series $\sum_{n=-\infty}^{\infty} f(n)$. Various methods exist. Residues can be used.

Theorem Let C_N be the square with vertices at the points $(N+\frac{1}{2})(1+i)$, $(N+\frac{1}{2})(1-i)$, $(N+\frac{1}{2})(-1-i)$, $(N+\frac{1}{2})(1-i)$, traversed counterclockwise. Let f be a function which is analytic except at the poles $z_1, \dots, z_m$ all of which are inside C_N. Suppose that on



C_N, f satisfies $|f(z)| \leq \frac{A}{|z|^k}$, $k>1$. (This guarantees convergence of the series), and f has no poles at $n = 0, \pm 1, \pm 2, \dots$ Then

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{k=1}^{m} \operatorname{Res}(\pi \cot(\pi z) f(z), z_k).$$

Proof The function $\cot(\pi z)$ has simple poles at $0, \pm 1, \pm 2$, etc. $\pi \cot(\pi z) f(z)$ has simple poles at $n \in \mathbb{Z}$. Then

$$\operatorname{Res}(\pi \cot(\pi z) f(z), n) = \lim_{z \to n} \frac{(z-n)\pi \cos(\pi z)\, f(z)}{\sin(\pi z)}$$

$$= \lim_{z \to n} \frac{\left(\pi \cos(\pi z) f(z) - (z-n)\pi^2 \sin(\pi z) f(z) + (z-n)\cos(\pi z)\, f'(z)\right)}{\pi \cos(\pi z)}$$

$$= f(n).$$

Let $S_N = \sum_{n=-N}^{N} f(n)$. By the residue theorem

$$\int_{C_N} \pi \cot(\pi z) f(z)\, dz = 2\pi i \left(S_N + \sum_{k=1}^{m} \mathrm{Res}(\pi\cot(\pi z) f(z), z_k) \right)$$

Let $N \to \infty$. We will show that

$$\int_{C_N} \pi \cot(\pi z) f(z)\, dz \to 0 \qquad (*)$$

Thus $\lim_{N\to\infty} 2\pi i \left(S_N + \sum_{k=1}^{m} \mathrm{Res}(\pi\cot(\pi z) f(z), z_k)\right)$

$$= 2\pi i \left(\sum_{n=-\infty}^{\infty} f(n) + \sum_{k=1}^{m} \mathrm{Res}(\pi\cot(\pi z) f(z), z_k)\right) = 0$$

Hence $\sum_{n=-\infty}^{\infty} f(n) = -\sum_{k=1}^{m} \mathrm{Res}(\pi\cot(\pi z) f(z), z_k)$

To show $(*)$, note length of the sides of the square are $2N+1$, $|\cot(\pi z)| \le M$, some constant $M>0$ on C_N. By the ML inequality

$$\left| \int_{C_N} \pi \cot(\pi z) f(z)\, dz \right| \le \frac{(8N+4)MA}{|z|^k} \le \frac{(8N+4)MA}{(N+1/2)^k} \to 0$$

as $N \to \infty$

Example Sum $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2}$. There are poles at $\pm ia$

$$\mathrm{Res}\left(\frac{\pi\cot(\pi z)}{z^2+a^2}, ia\right) = \lim_{z\to ia} \frac{(z-ia)\pi\cot(\pi z)}{(z-ia)(z+ia)}$$

$$= \lim_{z \to ia} \frac{\pi \cos(\pi i a)}{2ia \sin(\pi i a)} = \frac{\pi \cosh(\pi a)}{-2a\sinh(\pi a)}$$

Since $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, $\cos(\pi i a) = \frac{1}{2}(e^{-\pi a} + e^{\pi a})$

$= \cosh(\pi a)$

$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$, $\sin(\pi i a) = \frac{1}{2i}(e^{-\pi a} - e^{\pi a})$

$= i\sinh(\pi a)$

$$\operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2+a^2}, -ia\right) = \lim_{z\to -ia} \frac{(z+ia)\pi\cos(\pi z)}{\cancel{(z+ia)}(z-ia)\sin(\pi z)}$$

$$= \frac{\pi\cosh(\pi a)}{(-2ia)(-i)\sin(\pi a)} = -\frac{\pi\cosh(\pi a)}{2a\sinh(\pi a)}$$

$$\therefore \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = -\left(\frac{-\pi}{2a}\coth(\pi a) - \frac{\pi}{2a}\coth(\pi a)\right)$$

$$= \frac{\pi}{a}\coth(\pi a).$$

Also $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \sum_{n=-\infty}^{-1} \frac{1}{n^2+a^2} + \frac{1}{a^2} + \sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$

$$= 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + \frac{1}{a^2} = \frac{\pi}{a}\coth(\pi a)$$

or $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2}$

If f is even, $\displaystyle\sum_{n=1}^{\infty} f(n) = \frac{1}{2}\sum_{n=1}^{\infty} f(n) - \frac{1}{2}f(0)$.

<u>Theorem</u> Assume the same conditions as before. Then

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_{k=1}^{m} \operatorname{Res}(\pi\operatorname{cosec}(\pi z) f(z), z_k)$$

<u>Proof</u> Basically as before

Example Sum $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2}$ $\quad$ a is not an integer

Pole of order 2 at $-a$

$$\mathrm{Res}\left(\frac{\pi \operatorname{cosec}(\pi z)}{(z+a)^2}, -a\right) = \lim_{z\to -a} \frac{d}{dz}\left(\frac{(z+a)^2 \pi \operatorname{cosec}(\pi z)}{(z+a)^2}\right)$$

$$= \frac{-\pi^2 \cos(\pi a)}{\sin^2(\pi a)}$$

$$\therefore \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}.$$

Counting Zeroes. Suppose $f: S \subseteq \mathbb{C} \to \mathbb{C}$ is differentiable except at finitely many poles. Suppose none of the poles or zeroes of f lie on γ, contained in S. Let f have N zeroes inside γ (counted according to multiplicity). Let f have P poles, counted according to the order. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N - P$$

Proof Clearly $\displaystyle \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1} \mathrm{Res}\left(\frac{f'(z)}{f(z)}, z_k\right)$

If f has a zero of order k_1 at z_1 then f'/f has a pole of order 1 at z_1, with residue k_1.

We have $f(z) = (z - z_1)^{k_1}\phi(z)$

ϕ has no zeroes, ϕ is analytic

$$\text{So}\quad \frac{f'(z)}{f(z)} = \frac{k_1(z-z_1)^{k_1-1}\phi(z) + (z-z_1)^{k_1}\phi'(z)}{(z-z_1)^{k_1}\phi(z)}$$

$$= \frac{k_1}{z-z_1} + \underbrace{\frac{\phi'(z)}{\phi(z)}}_{\text{analytic}}$$

$\therefore$ $\frac{f'(z)}{f(z)}$ has a pole at z_1 with residue k_1

We can repeat this for every zero $z_2, \ldots z_n$. At z_2

$$\text{Res}\left(\frac{f'(z)}{f(z)}, z_2\right) = k_2, \quad \text{the multiplicity of } z_2$$

etc, So from the zeros

$$\sum \text{Res}\left(\frac{f'(z)}{f(z)}, z_k\right) = N = k_1 + \ldots + k_n$$

For the poles, write $f(z) = \frac{\psi(z)}{(z-z_p)^{n_p}}$

$$f'(z) = \frac{\psi'(z)}{(z-z_p)^{n_p}} - \frac{n_p\,\psi(z)}{(z-z_p)^{n_p+1}}$$

$$\frac{f'(z)}{f(z)} = \frac{\psi'(z)}{\psi(z)} - \frac{n_p}{z-z_p}.$$ Pole at z_p. residue $-n_p$

From the poles $z_p, z_{p+1}, \ldots, z_{p+m}$

$$\sum \text{Res}(f'(z)/f(z)) = -n_p - n_{p+1} - \ldots - n_{p+m}$$

$$= -P$$

So $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N - P$.

<u>Corollary</u>. If f has no poles

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N.$$

So we can calculate the number of zeroes from this integral.

The integral is usually computed numerically

<u>Example</u> $f(z) = z^n - 1$, $\gamma = 2e^{i\theta}$, $\theta \in [0, 2\pi]$

$$\frac{1}{2\pi i}\int_\gamma \frac{nz^{n-1}}{z^n - 1}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{2^n n i e^{ni\theta}}{2^n e^{ni\theta} - 1}\,d\theta$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{2^n n i e^{ni\theta} - n + n}{2^n e^{ni\theta} - 1}\,d\theta$$

$$= n + \underbrace{\frac{1}{2\pi i}\int_0^{2\pi} \frac{ni}{2^n e^{ni\theta} - 1}\,d\theta}_{=0}$$

$\therefore$ $f(z) = z^n - 1$ has n zeroes inside γ.

<u>Inversion of Laplace Transforms</u>

If f is integrable then the Laplace transform of f is

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

<u>Example</u> $f(t) = e^{at}$, $\int_0^\infty e^{at-st}\,dt$

$$= \int_0^\infty e^{-(s-a)t}\,dt = -\frac{e^{-(s-a)t}}{s-a}\Big|_0^\infty$$

$$= \frac{1}{s-a}$$

$$f(t) = \sin t = \frac{1}{2i}\left(e^{it} - e^{-it}\right)$$

$$F(s) = \frac{1}{2i}\int_0^\infty \left(e^{it} - e^{-it}\right) e^{-st}\,dt$$

$$= \frac{1}{2i}\left[\frac{1}{s-i} - \frac{1}{s+i}\right]$$

$$= \frac{1}{2i}\,\frac{2i}{s^2+1} = \frac{1}{s^2+1}$$

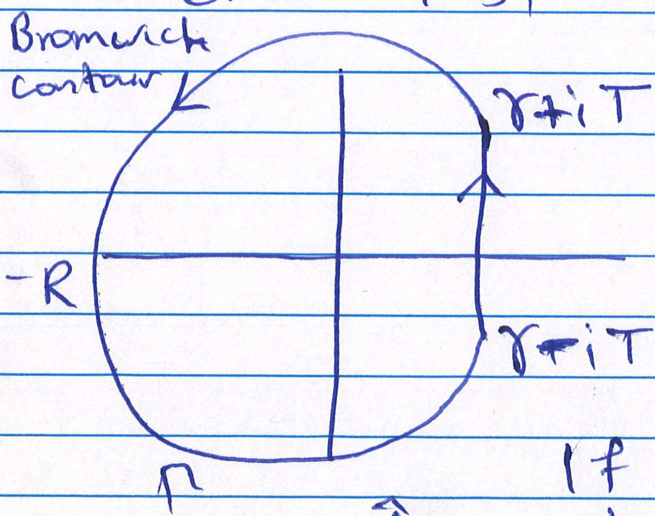
$f(t) = \cos t$, $\quad F(s) = \dfrac{s}{s^2+1}$.

Given $F(s)$, can we find $f(t)$?
This is the inverse Laplace transform

<u>Theorem</u> If f is differentiable, then

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} F(s)e^{st}\,ds = f(t)$$

where $F(s) = \int_0^\infty f(t)e^{-st}\,dt$



(we integrate around Γ. See below)
The proof uses the Fourier inversion theorem

If $\hat{f}(y) = \int_{-\infty}^{\infty} e^{-iyx} f(x)\,dx$
and $\hat{f}$ is integrable

$$F(s) = \int_0^\infty f(u) e^{-su}\, du$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(y) e^{iyx}\, dy.$$

Now $\lim\limits_{T\to\infty} \frac{1}{2\pi i} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds$

Put $s = \gamma + iy$, $ds = i\, dy$

Integral becomes $\lim\limits_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} e^{\gamma t + iyt} F(\gamma + iy)\, dy$

$$= \lim \frac{1}{2\pi} \int_{-T}^{T} e^{iyt} e^{\gamma t} \int_0^\infty e^{-(\gamma + iy)u} f(u)\, du$$

$$= \lim_{T\to\infty} \frac{e^{\gamma t}}{2\pi} \int_{-T}^{T} e^{iyt} \int_0^\infty [e^{-\gamma u} f(u)] e^{-iyu}\, du$$

Let $\tilde{f}(u) = \begin{cases} e^{-\gamma u} f(u) & u \geq 0 \\ 0 & u < 0 \end{cases}$

Then $\int_0^\infty e^{-\gamma u} f(u) e^{-iyu}\, du = \int_{-\infty}^{\infty} \tilde{f}(u) e^{-iyu}\, du$

So $\lim\limits_{T\to\infty} \frac{e^{\gamma t}}{2\pi} \int_{-T}^{T} e^{iyt} \int_{-\infty}^{\infty} \tilde{f}(u) e^{-iyu}\, du$

$= e^{\gamma t} \tilde{f}(t)$ By the Fourier inversion theorem.

$$= \begin{cases} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Example

$F(s) = \frac{s}{s^2+1}$ $\quad s = \pm i$ poles

$$\text{Res}\left(\frac{se^{st}}{s^2+1}, i\right) = \lim_{s\to i} \frac{(s-i)se^{st}}{(s-i)(s+i)} = \frac{ie^{it}}{2i}$$

$$\text{Res}\left(\frac{se^{st}}{s^2+1}, -i\right) = \lim_{s\to -i} \frac{(s+i)se^{st}}{(s+i)(s-i)} = \frac{e^{-it}}{2}.$$

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)\,ds = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$
$$= \cos t$$

We only have to sum the residues. The idea is to compute $\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)\,ds$ by integrating around Bromwich contour. As $T\to\infty$ the integral around the circular part can be shown to converge to zero, leaving the desired integral.