Differentiation Continued.

We define the derivative of $f: \mathbb{C} \to \mathbb{C}$ by

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}. \qquad (1)$$

This is the same definition as on $\mathbb{R}$. However there is a major difference. On $\mathbb{R}$ we can approach $x=0$ in only two directions. But in $\mathbb{C}$ there are infinitely many directions from which we can approach the origin. Thus the existence of the limit (1) is a stronger condition on f.

To get a condition on a function f which guarantees differentiability, we need some facts.

Definition Let $f: S \subseteq \mathbb{C} \to \mathbb{C}$ and let $z = x+iy$. Then we have

$$f(x+iy) = u(x,y) + i\,v(x,y)$$

and u is the real part of f and v is the imaginary part of f.

Example (1) $f(z) = z^2$. Then

$$f(x+iy) = (x+iy)^2$$

$$= x^2 - y^2 + 2ixy.$$

and $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$.

Example 2. $f(z) = e^z$

So $f(x+iy) = e^{x+iy} = e^x e^{iy}$

$$= e^x(\cos y + i \sin y)$$

$u(x,y) = e^x \cos y$, $v(x,y) = e^x \sin y$.

Observe that if $f(x+iy) = x^2 - y^2 + 2iyx$
Then

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$$

So $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Similarly if $f(x+iy) = e^x \cos y + i e^x \sin y$
$= u + iv$

Then $\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$

and $\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$.

These are the Cauchy Riemann equations.

<u>Theorem</u> If $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ and $f: S \subseteq \mathbb{C} \to \mathbb{C}$ and at some point $z_0 = x_0 + iy_0 \in S$, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist are continuous and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Then f is differentiable at z_0. Conversely, if f is differentiable at z_0, then the given partial derivatives exist and the C-R equations are satisfied at $z = x + iy$.

<u>Proof</u>

Assume that $f'(z)$ exists
Let

$$f(x+iy) = u(x,y) + iv(x,y)$$

We can let $h \to 0$ in infinitely many ways. Two obvious choices are along the real axis and along the imaginary axis and the limits must be equal.

First let $z = x+iy$, consider $z+h = x+h+iy$.

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{u(x+h,y)-u(x,y)}{h} + i\lim_{h\to 0} \frac{v(x+h,y)-v(x,y)}{h}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \qquad (A)$$

Now consider $z+ik$. This is a limit along imaginary axis

$$\lim_{k\to 0} \frac{f(z+ik)-f(z)}{ik} = \lim_{k\to 0} \frac{u(x,y+k)-u(x,y)}{ik} + i\lim_{k\to 0} \frac{v(x,y+k)-v(x,y)}{ik}$$

$$= -i \lim_{k\to 0} \frac{u(x,y+k)-u(x,y)}{k} + \lim_{k\to 0} \frac{v(x,y+k)-v(x,y)}{k}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \qquad (B)$$

Since the limit exists, (A) and (B) must be the same.

So $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$.

or $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

Remainder of proof is to follow

Example $f(z) = z^4$, $z = x+iy$.
Show that $f'(z)$ exists and find it

$$f(x+iy) = (x+iy)^4 = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$= x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3)$$

$\therefore$

$$u(x,y) = x^4 - 6x^2y^2 + y^4$$

$$v(x,y) = 4x^3y - 4xy^3$$

$$u_x = 4x^3 - 12xy^2$$

and

$v_y = 4x^3 - 12xy^2$. Thus $u_x = v_y$ all x, y.

$$u_y = -12x^2y + 4y^3$$

$v_x = 12x^2y - 4y^3 = -u_y$. all x, y.

Thus the CR equations are satisfied. And $f(z)$ is differentiable everywhere

Definition A function differentiable everywhere is said to be an entire function

Now $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$$= 4x^3 - 12xy^2 + i(12x^2y - 4y^3)$$

$$= 4(x^3 - 3xy^2 + i(3x^2y - y^3))$$

$$= 4(x+iy)^3 = 4z^3.$$

Example $f(z) = \sin z$.

Here $f(x+iy) = \sin(x+iy)$

$$= \sin x \cos(iy) + \sin(iy)\cos x$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

From Euler's formula

So $\sin(iy) = \dfrac{e^{-y} - e^{y}}{2i} = i\left(\dfrac{e^{y} - e^{-y}}{2}\right)$

$$= i\sinh y.$$

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh y.$$

Thus $\sin(x+iy) = \sin x \cosh y + i\cos x \sinh y$.

$u(x,y) = \sin x \cosh y, \quad v(x,y) = \cos x \sinh y.$

$\therefore u_x = \cos x \cosh y, \quad v_y = \cos x \cosh y$

$u_y = \sin x \sinh y, \quad v_x = -\sin x \sinh y$

$\therefore u_x = v_y$

$u_y = -v_x$

Thus f' exists.

$$f'(z) = u_x + i v_x$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$= \cos(x+iy) = \cos z$$

Since

$\cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)$

$= \cos x \cosh y - i \sin x \sinh y$

$$\therefore \frac{d}{dz} \sin z = \cos z$$

Example $f(z) = z$ $u(x,y) = x$, $v(x,y) = y$

$u_x = 1$, $v_y = 1$

$u_y = 0$, $v_x = 0$

So $u_x = v_y$, $v_x = -u_y$. So f' exists

What about $f(z) = \bar{z}$ $u = x$, $v = -y$

$u_x = 1$, $v_y = -1$

$\therefore u_x \neq v_y$

So f is not differentiable.

In fact if f is differentiable

$\overline{f(z)} = f(\bar{z})$ (Proof later)

If f is differentiable, $f(\bar{z})$ will not be differentiable.

A differentiable function $f: S \subseteq \mathbb{C} \to \mathbb{C}$ is said to be analytic.

On $\mathbb{R}$, a function f is analytic around x_0 if

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

around x_0. Question: Are these definitions different? No. We will see why later.

Some useful properties

Laplace's equation on $\mathbb{R}^2$ is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

In general $\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0$

If $f(z)$ is differentiable and $f(x+iy) = u(x,y) + iv(x,y)$, then u, v satisfy Laplace's equation. i.e. they are harmonic

Since $u_x = v_y$, $u_y = -v_x$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= -\frac{\partial^2 u}{\partial y^2}$$

So $\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0$. Similarly for v

<u>Example</u> $f(z) = z^2$

Then $f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy$

So

$$u = x^2 - y^2, \quad v(x,y) = 2xy$$

Then $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial u}{\partial y} = -2y$, $\frac{\partial^2 u}{\partial y^2} = -2$

So $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$

and $\frac{\partial^2 v}{\partial x^2} = 0$, $\frac{\partial^2 v}{\partial y^2} = 0$

For every positive integer n, $f(z) = z^n$ is analy[illegible]. So the real and imaginary parts are <u>harmonic</u> (ie satisfy Laplace's equation)

The real and imaginary parts are polynomials. They are called <u>harmonic</u> polynomials

If $f: \mathbb{R} \to \mathbb{R}$ is differentiable and $f' = 0$, then f is a constant. The same is true on $\mathbb{C}$.

<u>Proposition</u> Let $f: S \subseteq \mathbb{C} \to \mathbb{C}$. Suppose $f' = 0$ in S. Then f is a constant on S.

<u>Proof</u> $f'(z) = u_x + i v_x = 0$. So $u_x = v_x = 0$

But $u_x = v_y$, $u_y = -v_x$ $\therefore u_y = v_y = 0$

Thus u, v are constants. Hence f is constant

There are similar results

<u>Prop</u> If f is differentiable on $S \subseteq \mathbb{C}$, and any one of $|f|$, $Re(f)$, $Im(f)$ are constant, then f is constant

<u>Proof</u> We do one of them. If $f = u + iv$, $Re(f) = u$. Now $u_x = u_y = 0$. But $v_x = -u_y$, $v_y = u_x$. Thus v is a constant.

Given a function $u(x,y)$ an important question is can we find $v(x,y)$ such that $u(x,y) + iv(x,y) = f(x+iy)$ is differentiable. (we can do this the other way around too). Such a v is called the harmonic conjugate of u

We use the CR equations.

<u>Example</u> $u(x,y) = x^2 - y^2$. Then $u_x = 2x$, $u_y = -2y$. So $v_y = 2x$. Hence

$$v = 2xy + g(x)$$

But $v_x = 2y + g'(x) = 2y$. $\therefore g'(x) = 0$

So g is a constant

Thus $v = 2xy + C$

and $u + iv = x^2 - y^2 + 2ixy + iC$

$= z^2 + iC$.

Proof of the second half of the CR equations we need the following. If $u(x,y)$ is continuously differentiable we can find $A(h,k)$, $B(k,h)$ such that

$$u(x+h, y+k) = u(x,y) + h\left(\frac{\partial u}{\partial x} + A(h,k)\right) + k\left(\frac{\partial u}{\partial y} + B(h,k)\right)$$

and $A, B \to 0$ as $h, k \to 0$

Let $z = z_0 + h + ik$, or $z - z_0 = h + ik$.

Now we show that if the CR equations are satisfied, then f' exists.

Now $f(z) - f(z_0) = u(x_0+h, y_0+k) + iv(x_0+h, y_0+k) - u(x_0, y_0) - iv(x_0, y_0)$

$$= h\left(\frac{\partial u}{\partial x} + A(h,k)\right) + k\left(\frac{\partial u}{\partial y} + B(h,k)\right) + ih\left(\frac{\partial v}{\partial x} + A_1(h,k)\right) + ik\left(\frac{\partial v}{\partial y} + B_1(h,k)\right)$$

$$= (h+ik)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \theta z$$

$$= (z - z_0)(u_x + iv_x) + \theta z$$

θz is the error term. So

$$\frac{f(z) - f(z_0)}{z - z_0} = u_x + iv_x + \frac{\theta z}{z - z_0}$$

Hence $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) = u_x + iv_x$

$\frac{\theta z}{z - z_0} \to 0$ as $z \to z_0$, since $A, B, A_1, B_1 \to 0$ as $h, k \to 0$

Hence f' exists.