Integration in the Complex Plane

We start with an obvious definition

If $h: [a,b] \to \mathbb{C}$, with

$$h(t) = h_1(t) + i\,h_2(t)$$

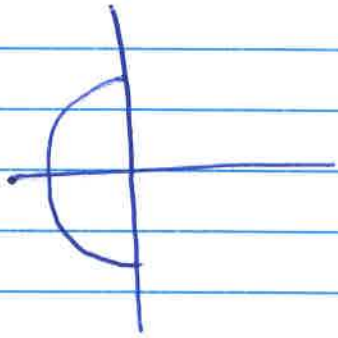
Then
$$\int_a^b h(t)\,dt = \int_a^b h_1(t)\,dt + i\int_a^b h_2(t)\,dt$$

Shortly we will use this to define a contour.

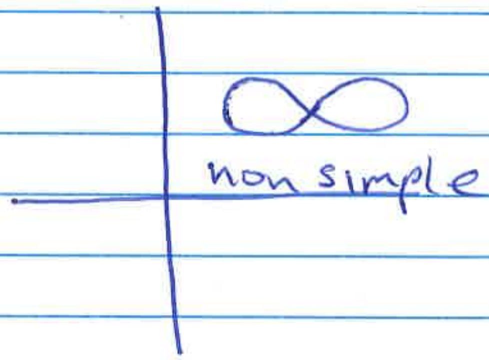
<u>Definition</u> A simple, smooth contour between two points $z_0, z_1 \in \mathbb{C}$ is a parameterised curve $\gamma: [a,b] \to \mathbb{C}$ such that

(1) $\gamma'(t)$ exists for all $t \in (a,b)$.
(2) $\gamma(a) = z_0$, $\gamma(b) = z_1$
(3) γ does not cross itself.

<u>Example</u>



simple



<u>Definition</u> Let $f: D \subseteq \mathbb{C} \to \mathbb{C}$ be continuous. Suppose $z_0, z_1 \in D$. Let $\gamma: [a,b] \to D$ be a simple, smooth contour with $\gamma(a) = z_0$, $\gamma(b) = z_1$. Then the contour integral of f over γ is

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

The idea comes from physics. We have the idea of work. If a force $\underset{\sim}{F}$ acts on a particle over a distance $\underset{\sim}{d}$, the work done is

$$W = \underset{\sim}{F} \cdot \underset{\sim}{d}$$

Let the force act at time t at the point $\underset{\sim}{x}(t)$, suppose the force at this point is $\underset{\sim}{F}(\underset{\sim}{x}(t))$. The total work done in moving from $\underset{\sim}{x}(t)$ to $\underset{\sim}{x}(t+\Delta t)$ is

$$W(t+\Delta t) - W(t) \simeq \underset{\sim}{F}(\underset{\sim}{x}(t)) \cdot [\underset{\sim}{x}(t+\Delta t) - \underset{\sim}{x}(t)]$$

Thus

$$\frac{W(t+\Delta t) - W(t)}{\Delta t} = \underset{\sim}{F}(\underset{\sim}{x}(t)) \cdot \frac{(\underset{\sim}{x}(t+\Delta t) - \underset{\sim}{x}(t))}{\Delta t}$$

Let $\Delta t \to 0$, then

$$\frac{dW}{dt} = \underset{\sim}{F}(\underset{\sim}{x}(t)) \cdot \frac{d\underset{\sim}{x}}{dt}$$

$$\left[\begin{array}{l} \underset{\sim}{x}(t) = (x_1(t), \dots x_n(t)) \\ \frac{d}{dt}\underset{\sim}{x}(t) = (x_1'(t), \dots, x_n'(t)) \end{array} \right]$$

So $\displaystyle\int_a^b \frac{dW}{dt}\,dt = W(b) - W(a) = \int_a^b \underset{\sim}{F}(\underset{\sim}{x}(t)) \cdot x'(t)\,dt$

This is called a line integral. It is the basis of the definition of a contour integral.

<u>Example</u> $f(z) = z^2$. $z_0 = 0$, $z_1 = 1+i$

Take $\gamma(t) = (1+i)t$, $t \in [0,1]$

$$\gamma'(t) = 1+i$$

$$f(\gamma(t)) = ((1+i)t)^2 = (1+i)^2 t^2$$

So

$$\int_\gamma f(z)\,dz = \int_0^1 (1+i)^2 t^2 (1+i)\,dt = (1+i)^3 \int_0^1 t^2\,dt$$

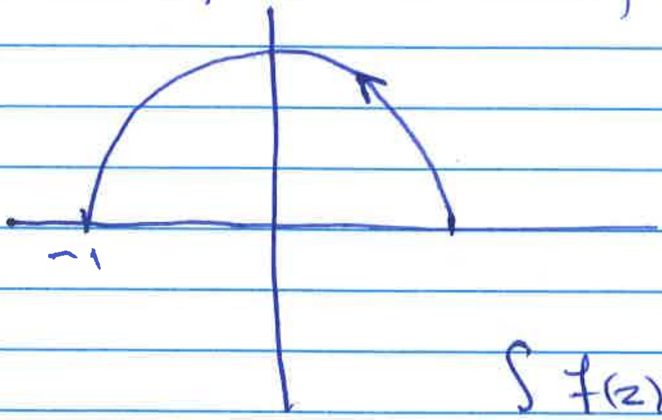
$$= (1+i)^3 \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}(1+i)^3.$$

$$= -\frac{2}{3} + \frac{2}{3}i$$

<u>Example</u> $f(z) = z^2 \quad \gamma(t) = t^2 + it, \ t \in [0,1]$

Then $$\int_\gamma f(z)\,dz = \int_0^1 (t^2+it)^2 (2t+i)\,dt$$
$$= \int_0^1 (t^4 + 2it^3 - t^2)(2t+i)\,dt$$
$$= \int_0^1 (2t^5 + 4it^4 - 2t^3 + it^4 - 2t^3 - it^2)\,dt$$
$$= \left[\frac{1}{3}t^6 + \frac{4i}{5}t^5 - \frac{1}{2}t^4 + \frac{it^5}{5} - \frac{1}{2}t^4 - \frac{it^3}{3}\right]_0^1$$
$$= \frac{1}{3} + \frac{4i}{5} - \frac{1}{2} + \frac{i}{5} - \frac{1}{2} - \frac{i}{3} = -\frac{2}{3} + \frac{2}{3}i$$

<u>Example</u> $f(z) = z^2, \ \gamma(t) = e^{it}, \ t \in [0,\pi]$



This curve moves counter clockwise. This is the convention.

$\gamma(0) = 1, \ \gamma(\pi) = -1$

$$\int_\gamma f(z)\,dz = \int_0^\pi (e^{it})^2 \, i e^{it}\,dt$$
$$= i\int_0^\pi e^{3it}\,dt = \left[\cancel{i}\frac{e^{3it}}{3\cancel{i}}\right]_0^\pi$$
$$= \frac{1}{3}(-1-1) = -\frac{2}{3}.$$

Recall that $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$. There are other inequalities. eg. Cauchy-Schwartz. For contour integrals we also have inequalities

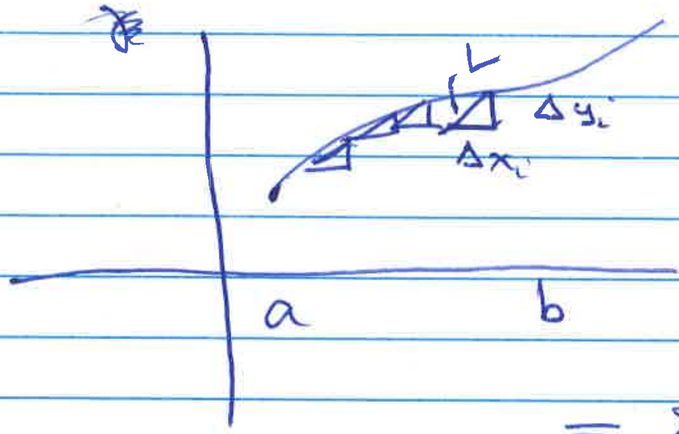
Proposition Let $\gamma : [a,b] \to \mathbb{C}$, be a simple closed contour. The length of γ, $L(\gamma)$ between a and b is

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt$$

Proof The length of a curve $(x(t), y(t))$, $t \in [a,b]$ is

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

The result follows from this by taking γ!



$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

length of curve is roughly

$$\sum \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \sum \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

as $\Delta x \to 0$

$$\longrightarrow \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

So the length of the curve $y = f(x)$ between $y(a)$, $y(b)$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

eg. $y = \sqrt{1-x^2}$, $-1 \le x \le 1$ $\quad \frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$

$$L = \int_{-1}^1 \sqrt{1 + \frac{x^2}{1-x^2}}\, dx = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\, dx = \pi$$

(put $x = \sin\theta$) (check).

The formula for a parameterised curve $(x(t), y(t))$ comes from this.

If $\gamma(t) = x(t) + iy(t)$. Then γ

$$\gamma'(t) = x'(t) + iy'(t)$$

So $|\gamma'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$

Thus $\int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

= length of γ.

<u>Lemma</u> (The ML inequality). Suppose that $f: D \subseteq \mathbb{C} \to \mathbb{C}$ has a maximum modulus of $M > 0$. i.e. $|f(z)| \leq M$, on the contour γ. Suppose $L(\gamma) = L$.

Then $\left|\int_\gamma f(z)\,dz\right| \leq ML$.

<u>Proof</u>

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right|$$
$$\leq \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt$$
$$\leq \max_{t\in[a,b]} |f(\gamma(t))| \int_a^b |\gamma'(t)|\,dt$$
$$= ML$$

<u>The Fundamental theorem of contour integration</u>

Suppose that $F' = f$, $D \subseteq \mathbb{C}$ is open and $f: D \to \mathbb{C}$ is continuous and γ is a smooth, simple contour in D with

end points z_0, z_1. Then
$$\int_\gamma f(z)\,dz = F(z_1) - F(z_0).$$

Proof

Let $z = \gamma(t)$ $\quad \frac{d}{dt}F(\gamma(t)) = \frac{dF}{dz}\frac{dz}{dt}$

$$= f(\gamma(t))\gamma'(t)$$

So
$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$
$$= \int_a^b \frac{dF}{dt}\,dt$$
$$= F(\gamma(b)) - F(\gamma(a))$$
$$= F(z_1) - F(z_0)$$
Since $\gamma(a) = z_0$, $\gamma(b) = z_1$

Example $f(z) = z^2$. $\quad \gamma$ any contour from z_0 to z_1, $z_0 = 0$, $z_1 = 1+i$
$$\int_\gamma f(z)dz = \int_0^{1+i} f(z)dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{(1+i)^3}{3}$$
which is the same answer as before

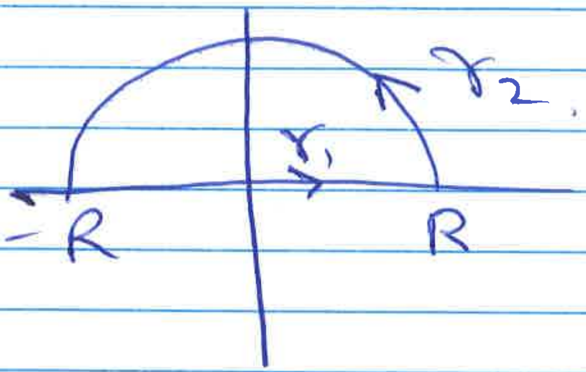
Example $f(z) = ze^z$, $\gamma(t) = t + it^2$, $t \in [0,1]$
$z_0 = 0$, $z_1 = 1+i$.
$$\int_\gamma f(z)dz = \int_0^1 (t+it^2)e^{t+it^2}(1+2it)\,dt. \quad \underline{\text{OR}}$$
$$\frac{d}{dz}(z-1)e^z = ze^z$$
So
$$\int_\gamma f(z)\,dz = \int_0^{1+i} ze^z dz = \left[(z-1)e^z\right]_0^{1+i} = 1 + ie^{1+i}.$$

Integration along systems of contours

Example $\gamma_1(t) = t \quad -R \leq t \leq R$

$\gamma_2(t) = Re^{it}, \quad t \in [0, \pi]$

The contour $\gamma = \gamma_1 + \gamma_2$ is



We DO NOT ADD the functions

$\gamma_1 + \gamma_2$ means: travel along γ_1 then along γ_2.

Definition If $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f + \cdots + \int_{\gamma_n} f$$