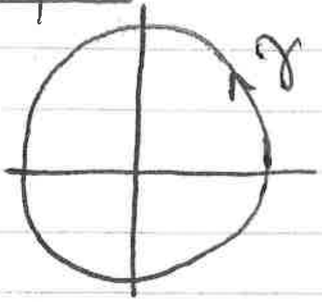
Applications of the Fundamental Theorem

In this subject we are primarily interested in integration around closed paths.

<u>Example</u> $f(z) = z^2$, $\gamma(t) = e^{it}$, $t \in [0, 2\pi)$.



$$\text{Now } \int_\gamma f = \int_0^{2\pi} (e^{it})^2 \, i e^{it} \, dt$$

$$= \int_0^{2\pi} i e^{3it} \, dt = \left[i \frac{1}{3i} e^{3it} \right]_0^{2\pi}$$

$$= \frac{1}{3}(e^{6\pi i} - e^0) = 0$$

This result is not unexpected and in fact $\int_\gamma f = 0$ for any differentiable f, smooth, simple closed γ. This is Cauchy's Theorem. A preliminary result is

<u>Theorem</u>

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a continuous differentiable function. Then the following are equivalent

(1) There is an $F : D \to \mathbb{C}$ such that $F' = f$ in D

(2) $\int_\gamma f = 0$ for every closed, simple smooth contour $\gamma \in D$

(3) $\int_\gamma f$ depends only on the end points of γ.

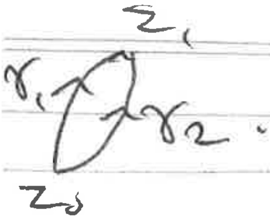
<u>Proof</u> (1) $\Rightarrow$ (2) Let γ be a closed contour which goes from z_0 to z_0 i.e. $\gamma : [a, b] \to \mathbb{C}$ and $\gamma(a) = \gamma(b) = z_0$

Then by the Fundamental Theorem

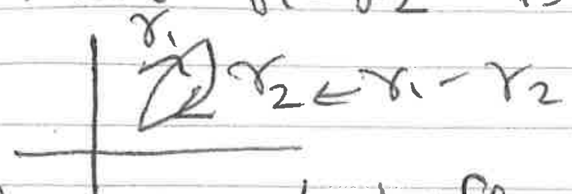
$$\int_\gamma f \stackrel{\text{def}}{=} \int_a^b f(\gamma(t))\gamma'(t)\,dt = F(\gamma(a)) - F(\gamma(b))$$

$$= F(z_0) - F(z_0) = 0$$

(2) $\Rightarrow$ (3) Let γ_1, γ_2 both go from z_0 to z_1.



Then $\gamma_1 - \gamma_2$ is closed



So $\int_\gamma f = 0$, but $\int_\gamma f = \int_{\gamma_1} f - \int_{\gamma_2} f = 0$

$$\therefore \int_{\gamma_1} f = \int_{\gamma_2} f.$$

(3) $\Rightarrow$ (1). We construct F. Pick $z_0 \in D$ and let z_1 be another point in D

Let $F(z_1) = \int_\gamma f$, where γ connects z_0, z_1

Now pick h and let $z_1 + h$ be joined to z_1 by $\lambda(t) = z_1 + ht$.

Then $F(z_1+h) = \int_\gamma f + \int_\lambda f$

and $F(z_1+h) - F(z_1) = \int_\gamma f + \int_\lambda f - \int_\gamma f = \int_\lambda f$

Hence

$$\frac{F(z_1+h) - F(z_1)}{h} = \frac{1}{h}\int_\lambda f$$

Now $f(z_1) = \frac{1}{h}\int_\lambda f(z_1)\,dz$

So

$$\frac{F(z_1+h) - F(z_1)}{h} - f(z_1) = \frac{1}{h}\int_\lambda (f(z) - f(z_1))\,dz$$

But f is continuous. So given $\varepsilon > 0$ $\exists \delta > 0$ st.

$|z - z_1| < \delta \Rightarrow |f(z) - f(z_1)| < \varepsilon$.

Choose $|h| < \delta$, $\Rightarrow \left|\frac{f(z) - f(z_1)}{h}\right| < \frac{\varepsilon}{|h|}$.

By the ML inequality

$$\left|\frac{F(z_1+h) - F(z_1)}{h} - f(z_1)\right| = \left|\frac{1}{h}\int_\lambda (f(z) - f(z_1))\,dz\right|$$

$$< |h| \frac{\varepsilon}{|h|} = \varepsilon \quad \therefore F' = f.$$