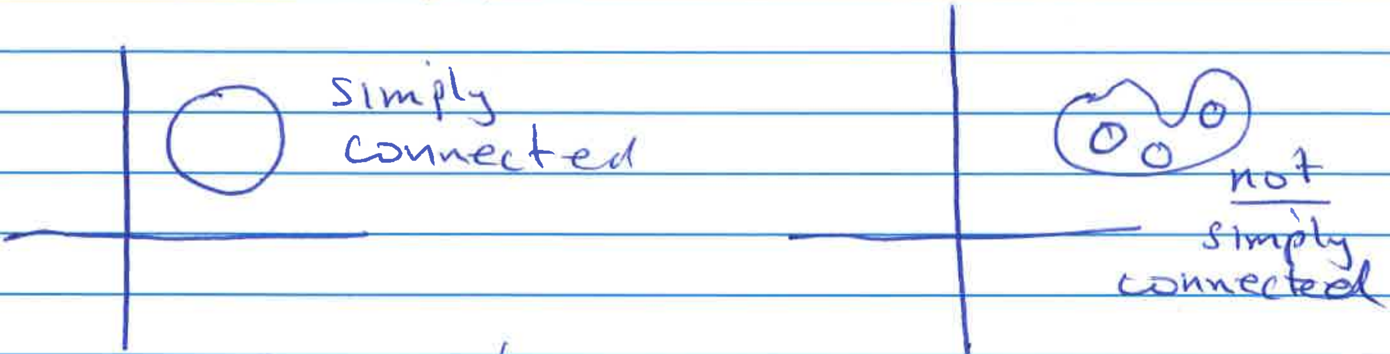
Cauchy's Theorem Let D be a simply connected domain in $\mathbb{C}$. Let f be differentiable in D. Suppose that γ is a simple, smooth closed curve in D. Then

$$\int_\gamma f(z)\,dz = 0.$$

Proof This needs new material.

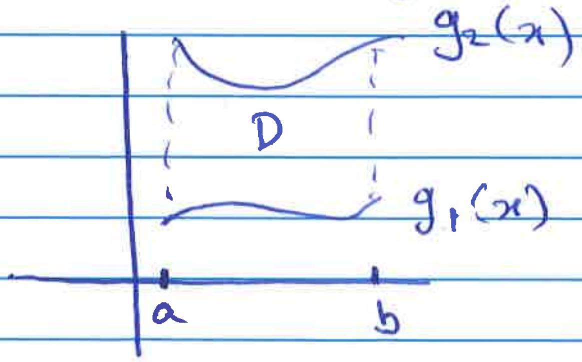
Definition D is simply connected if it has no holes.



To prove Cauchy's Theorem we need Green's Theorem. Green's Theorem in the plane is a statement about double integrals.

Let

$$D = \{(x,y) \subseteq \mathbb{R}^2,\ a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$



Then $\iint_D f(x,y)\,dy\,dx$

$$\stackrel{\text{def}}{=} \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$

This is an iterated integral.

or

$$D = \{(x,y) \subseteq \mathbb{R}^2,\ c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

Example $I = \int_1^4 \int_{-1}^2 (2x + 6x^2y)\,dy\,dx$

$$= \int_1^4 \left(\int_{-1}^2 (2x+6x^2y)\,dy \right) dx$$

$$= \int_1^4 \left[2xy + 3x^2y^2 \right]_{-1}^{2} dx$$

$$= \int_1^4 (6x + 9x^2)\,dx = \left[3x^2 + 3x^3 \right]_1^4$$

$$= 234.$$

Exercise $\int_{-1}^2 \int_1^4 (2x + 6x^2y)\,dx\,dy = 234$

In general if f is continuous on D then

$$\iint_D f(x,y)\,dx\,dy = \iint_D f(y,x)\,dy\,dx$$

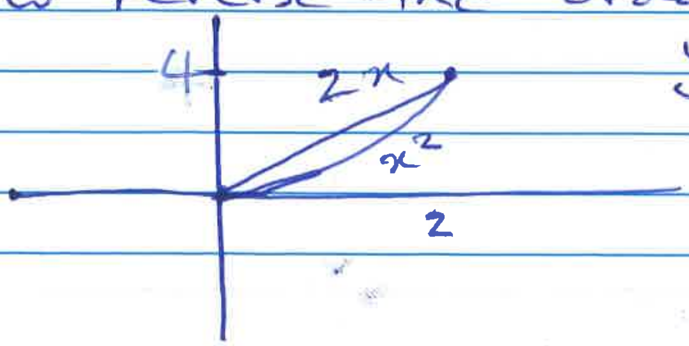
if $z = f(x,y)$, and f is positive then $\iint_D f$ is the volume between D and the surface $z = f(x,y)$

Example Evaluate $I = \int_0^2 \int_{x^2}^{2x} (x^3 + 4y)\,dy\,dx$

$$= \int_0^2 \left(\int_{x^2}^{2x} (x^3 + 4y)\,dy \right) dx = \int_0^2 \left[x^3y + 2y^2 \right]_{x^2}^{2x} dx$$

$$= \int_0^2 (8x^2 - x^5) = \left[\frac{8}{3}x^3 - \frac{1}{6}x^6 \right]_0^2 = \frac{32}{3}.$$

Now reverse the order of integration



$y = 2x, \quad x = \frac{1}{2}y$

$y = x^2, \quad x = \sqrt{y}$

So $\quad 0 \le y \le 4$

$\frac{1}{2}y \le x \le \sqrt{y}.$

$$I = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^3+4y)\,dx\,dy$$
$$= \int_0^4 \left[\tfrac{1}{4}x^4 + 4xy\right]_{y/2}^{\sqrt{y}} dy$$
$$= \int_0^4 \left(4y^{3/2} - y^4/64 - \tfrac{7}{4}y^2\right) dy = \frac{32}{3}$$

as before

Changes of variables

(I) Polar coordinates, we can let

$x = r\cos\theta,\ y = r\sin\theta,\quad 0 \le a \le r \le b$

$0 \le \theta \le 2\pi$

Then $dx\,dy = r\,dr\,d\theta$

In general if $x = x(u,v),\ y = y(u,v)$

Then

$$dx\,dy = \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right\| = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right|$$

This is called the Jacobian determinant

Named for Carl Jacobi

$$J(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$ is called the Jacobian

What does it mean?

If $f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$. Then

$$L(h) = \begin{pmatrix} f_1(a,b) \\ f_2(a,b) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x}(a,b) & \frac{\partial f_1}{\partial y}(a,b) \\ \frac{\partial f_2}{\partial x}(a,b) & \frac{\partial f_2}{\partial y}(a,b) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

Is the best linear approximation to $f(a+h)$

$= \begin{pmatrix} f_1(a+h_1, b+h_2) \\ f_2(a+h_1, b+h_2) \end{pmatrix}$ at (a,b) when $\|h\|$ is small

Example

$x = r\cos\theta, \quad y = r\sin\theta$

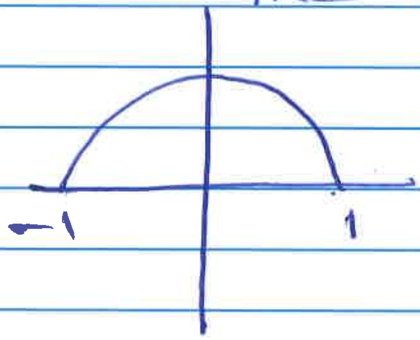
$$J(r,\theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det J = r\cos^2 + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta)$$
$$= r$$

$$\therefore \; dx\,dy = r\,dr\,d\theta$$

Example $f(x,y) = \sqrt{x^2+y^2}$. Calculate

$\iint_D f\,dx\,dy$, D is the semicircle of radius 1, centered at the origin



$$0 \le r \le 1$$
$$0 \le \theta \le \pi$$

$$\text{So } \iint_D f = \int_0^1 \int_0^\pi r\,r\,d\theta\,dr$$
$$= \int_0^1 r^2 \left(\int_0^\pi d\theta \right) dr$$
$$= \pi \int_0^1 r^2\,dr$$
$$= \pi \frac{r^3}{3} \Big|_0^1 = \frac{\pi}{3}.$$

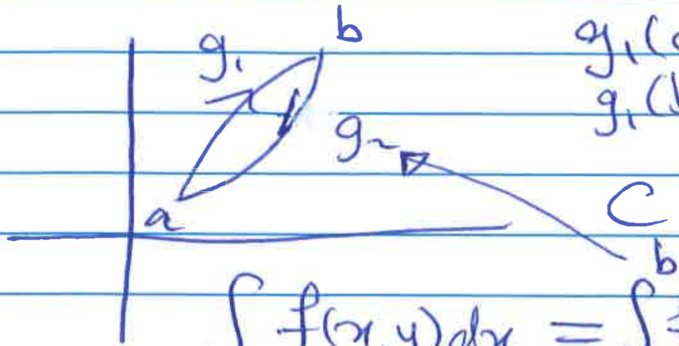
Line Integrals Given $f(x,y)$ we can integrate over a curve C in the following fashion. Let C be given by $g(x)$ $a \le x \le b$. Then

$$\int_C f(x,y)\,dx = \int_a^b f(x, g(x))\,dx$$

Example $f(x,y) = y^2$, $y = \cos x$, $x \in [0, \pi/2]$

$$\int_C f = \int_0^{\pi/2} \cos^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2x))\,dx = \frac{\pi}{4}$$

For closed curves, we think of two distinct curves g_1, g_2



$g_1(a) = g_2(b)$
$g_1(b) = g_2(a)$

C is the closed loop

$$\int_C f(x,y)\,dx = \int_a^b f(x, g_1(x))\,dx - \int_a^b f(x, g_2(x))\,dx$$

<u>Theorem</u> (Green's Theorem). Let D be a simply connected region in $\mathbb{R}^2$. Let C be its piecewise smooth boundary, which is traversed counterclockwise. Let P, Q be continuously differentiable in x, y in D. Then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

<u>Note</u> $\oint_C$ means it is an integral around a closed curve.

<u>Proof</u> Suppose $D = \{(x,y),\ a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

or $D = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

Then $$\iint_D \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\right) dx$$

$$= -\int_a^b \left[P(x, g_2(x)) - P(x, g_1(x))\right] dx$$

$$= \oint_C P\,dx$$

$$\therefore \iint \left(-\frac{\partial P}{\partial y}\right) dx\,dy = \oint_C P\,dx$$

Similarly $\iint_D \frac{\partial Q}{\partial x}\, dx\,dy = \int_C Q\,dy$.

Add these

<u>Example</u> $\oint_C (xy\,dx + (x-y)\,dy)$ where C is the boundary of the rectangle

$D = \{(x,y) : 0 \le x \le 1,\ 1 \le y \le 3\}$

$P = xy$, $Q = x-y$.

$\frac{\partial P}{\partial y} = x$, $\frac{\partial Q}{\partial x} = 1$.

$$\therefore \oint_C (xy\,dx + (x-y)\,dy) = \int_0^1\int_1^3 (1-x)\,dy\,dx$$

$$= \int_0^1 (1-x)y\Big|_1^3\,dx$$

$$= \int_0^1 2(1-x)\,dx = 1.$$

<u>Example</u> Calculate

$$\oint_C \underbrace{(x^3+y^3)}_{P}\,dx + \underbrace{(2y^3-x^3)}_{Q}\,dy,$$ where

C is the unit circle. $\frac{\partial P}{\partial y} = 3y^2$, $\frac{\partial Q}{\partial x} = -3x^2$

So $\oint_C \cdots = -\int_D 3(x^2+y^2)\,dx\,dy$ $\qquad x^2+y^2=r^2$

$x = r\cos\theta$

$y = r\sin\theta$

$$= -3\int_0^{2\pi}\int_0^1 r^2 . r\,dr\,d\theta$$

$$= -3\int_0^{2\pi}\left(\int_0^1 r^3\,dr\right)d\theta = -3 . 2\pi . \frac{1}{4}$$

$$= \frac{-3\pi}{2}.$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -3(x^2+y^2)$$

Proof of Cauchy's Theorem Let $f = u + iv$

$z = x + iy$, $dz = dx + i\,dy$

Then $\int_\gamma f(z)\,dz = \int_\gamma (u + iv)(dx + i\,dy)$

$$= \int_\gamma u\,dx - v\,dy$$

$$+ i\int_\gamma v\,dx + u\,dy.$$

$$= \int_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy,$$ D is the region enclosed by γ.

$$= \int_D \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy$$

$= 0$ by the CR equations.

And $\int_\gamma v\,dx + u\,dy = \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$

$= 0$ by the CR equations. Thus $\int_\gamma f(z)\,dz = 0$

Evaluate $\int_0^\infty \cos(t^2)\,dt$ and $\int_0^\infty \sin(t^2)\,dt$