Cauchy Integral Formula Let f be differentiable in open set $\Omega \subseteq \mathbb{C}$, which contains a closed disc $D_R = \{z \in \mathbb{C} : |z - z_0| \leq R\}$. If C_R is the boundary of D_R, traversed counterclockwise, then for $|z - z_0| < R$,

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\xi)}{\xi - z} \, d\xi$$

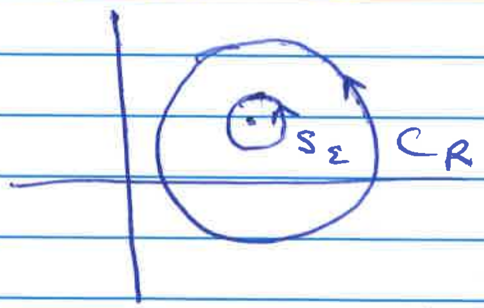
Note $\frac{1}{2\pi i} \frac{1}{\xi - z}$ is called the Cauchy kernel.

If $(Tf)(x) = \int_a^b K(x,y) f(y)\, dy$. K is called a kernel.

The Cauchy kernel is a reproducing kernel i.e. it gives f back.

Proof This follows from Cauchy's Theorem.

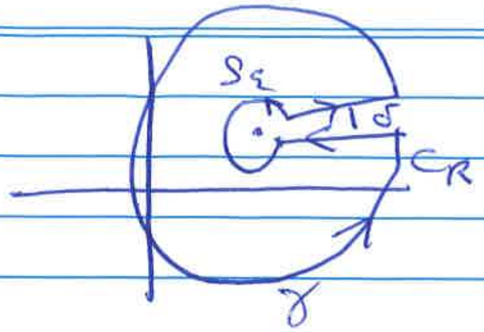
Let $F(\xi) = \frac{f(\xi) - f(z)}{\xi - z}$. This is differentiable in $D_R - \{z\}$. Consider a circle S_ε of radius ε about the point z. S_ε is contained in C_R.



The key to the proof is the fact that

$$\int_{C_R} F(\xi)\, d\xi = \int_{S_\varepsilon} F(\xi)\, d\xi$$

(Tute exercise. It is a consequence of Cauchy's Theorem)



$\int_\gamma F = 0$

let $\delta \to 0$

$\lim_{\xi \to z} F(\xi) = f'(z)$. f is differentable. Thus F is bounded inside S_ε. There is an $M > 0$, such that $|F(\xi)| \leq M$ for all ξ. The length of $S_\varepsilon = 2\pi\varepsilon$.

Thus $$\left| \int_{S_\varepsilon} F(\xi)\, d\xi \right| \leq 2\pi M \varepsilon \quad \text{(ML inequality)}$$

Since $$\int_{C_R} F = \int_{S_\varepsilon} F$$

Then $$\left| \int_{C_R} F(\xi)\, d\xi \right| \leq 2\pi M \varepsilon, \quad \text{For all } \varepsilon > 0.$$

Hence $$\int_{C_R} F(\xi)\, d\xi = 0$$

Or $$\int_{C_R} \frac{f(\xi) - f(z)}{\xi - z}\, d\xi = 0$$

Hence $$\int_{C_R} \frac{f(z)}{\xi - z}\, d\xi = \int_{C_R} \frac{f(\xi)}{\xi - z}\, d\xi$$

Put parameterise C_R by $\gamma(t) = z + Re^{it}$, $t \in [0, 2\pi)$

So $$\int_{C_R} \frac{f(z)}{\xi - z}\, d\xi = f(z) \int_0^{2\pi} \frac{1}{z + Re^{it} - z}\, iRe^{it}\, dt$$
$$= f(z) \int_0^{2\pi} i\, dt = 2\pi i f(z).$$

Therefore $f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\xi)}{\xi - z} d\xi$

<u>Example</u> $f(z) = e^z$, $z = \pi i$, C_R is circle of radius 1 centered at πi

$$\int_{C_R} \frac{e^\xi}{\xi - \pi i} d\xi = 2\pi i e^{\pi i} = -2\pi i$$

<u>Corollary</u> If f is differentiable in an open set D, then f has infinitely many derivatives, and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$$

where $C \subset D$ is a circle. C is traversed counterclockwise

<u>Proof</u> Idea is that $\frac{1}{\xi - z}$ is infinitely differentiable with respect to $z \neq \xi$

So $$f^{(n-1)}(z) = \frac{1}{2\pi i} \int_C \frac{d^{n-1}}{dz^{n-1}} \frac{f(\xi)}{(\xi - z)} d\xi$$
$$= \frac{(n-1)!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^n} d\xi$$

Let us assume this is true. Then

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h}$$
$$= \frac{(n-1)!}{2\pi i} \int_C f(\xi) \frac{1}{h} \left[\frac{1}{(\xi - z - h)^n} - \frac{1}{(\xi - z)^n} \right] d\xi$$

$$f^{(n)}(z) = \lim_{h \to 0} \frac{(n-1)!}{2\pi i} \int_C f(\xi) \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{(\xi - z - h)^n} - \frac{1}{(\xi - z)^n} \right] d\xi$$

$$= \frac{(n-1)!}{2\pi i} \int_C f(\zeta) \frac{d}{dz} \frac{1}{(\zeta - z)^n} d\zeta$$

$$= \frac{(n-1)!\, n}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

$$\therefore \quad f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

follows for all n by induction.

Theorem Let f be a differentiable function in an open set D. If C is a disc centered at z_0 with closure contained in D then for $z \in C$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n = \frac{1}{n!} f^{(n)}(z_0)$

Proof $f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$

Observe that

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)}$$

$$= \frac{1}{(\zeta - z_0)} \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} . \text{ Note } \left|\frac{z - z_0}{\zeta - z_0}\right| < 1$$

$$= \frac{1}{\zeta - z_0} \left(1 + \frac{(z - z_0)}{(\zeta - z_0)} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \left(\frac{z - z_0}{\zeta - z_0}\right)^3 + \cdots \right)$$

So

$$f(z) = \frac{1}{2\pi i} \int_C f(\zeta) \frac{1}{\zeta - z_0} \left(1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \cdots \right) d\zeta$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \frac{f(\xi)(z-z_0)^n}{(\xi - z_0)^{n+1}} d\xi$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i} (z-z_0)^n \int_C \frac{f(\xi)}{(\xi-z_0)^{n+1}} d\xi$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n$$

This means that the only differentiable functions in the complex plane are those equal to their Taylor series expansion. If f does not equal some Taylor series it has no derivative

Corallary The Cauchy inequalities.
Suppose that f is differentiable in an open set that contains the closure of an open disc D of radius R centered at z_0. Let C be the boundary of D.
Then
$$|f^{(n)}(z_0)| \leq \frac{n! \, \|f\|_C}{R^n}$$

$$\|f\|_C = \sup_{z \in C} |f(z)| \quad \left(= \max_{z\in C} |f(z)|\right)$$

Proof Parameterise C by $c(t) = z_0 + Re^{it}$ $t \in [0, 2\pi]$. Then

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right|$$

$$= \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{(Re^{it})^{n+1}} iRe^{it} dt \right|$$

$$= \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^n} dt \right|$$

$$\leq \frac{n!}{2\pi} \|f\|_c \int_0^{2\pi} \frac{dt}{R^n} = \frac{n!\, \|f\|_c}{R^n}$$

Liouville's Theorem If f is an entire function, which is bounded, then f is a contant.

Proof. Take $n=1$, suppose f is bounded on every disc D_R, (for all R). Pick $z_0 \in D$

Then $|f'(z_0)| \leq \frac{M}{R}$ M fixed, but R can be as large as we like

$\therefore f'(z_0) = 0$ for any $z_0 \in D$.

Thus f is a constant

The fundamental Theorem of Algebra Every polynomial of degree n has n roots, counted by multiplicity.

Proof Suppose $P_n(z)$ has no zeros Then $\frac{1}{P_n(z)}$ is entire and it is also bounded, since $\lim_{|z|\to\infty} P_n(z) = 0$

But this contradicts Liouville's Theorem So $P_n(z)$ has a zero z_1. But then $P_n(z) = (z - z_1) P_{n-1}(z)$. Apply the same argument to $P_{n-1}(z)$ to get a second root z_2. etc.