Complex Analysis Tutorial Nine Solutions.

Question One. We have the integral

$$\int_0^\infty \frac{x^{1/4}}{x^2+16}dx = \int_0^\infty \frac{4u^4}{u^8+16}du \qquad = \frac{1}{2}\int_{-\infty}^\infty \frac{4u^4}{u^8+16}du.$$

This is just an application of the residue theorem. There are four simple poles above the axis. These are solutions of $z^8 = -16$ and are $z_1 = \sqrt{2}e^{\frac{\pi i}{8}}, z_2 = \sqrt{2}e^{\frac{3\pi i}{8}}, z_3 = \sqrt{2}e^{\frac{5\pi i}{8}}, z_4 = \sqrt{2}e^{\frac{7\pi i}{8}}$. We have for $f(z) = \frac{4z^4}{z^4+16}$,

$$\begin{aligned}
\text{Res}(f(z), \sqrt{2}e^{\frac{\pi i}{8}}) &= \frac{1}{4\sqrt{2}i}e^{\frac{\pi i}{8}},\\
\text{Res}(f(z), \sqrt{2}e^{\frac{3\pi i}{8}}) &= -\frac{1}{4\sqrt{2}}e^{-\frac{\pi i}{8}},\\
\text{Res}(f(z), \sqrt{2}e^{\frac{5\pi i}{8}}) &= -\frac{1}{4\sqrt{2}}e^{-\frac{\pi i}{8}},\\
\text{Res}(f(z), \sqrt{2}e^{\frac{7\pi i}{8}}) &= \frac{1}{4\sqrt{2}i}e^{\frac{\pi i}{8}}.
\end{aligned}$$

We then obtain after an elementary application of Euler's formula

$$\int_0^\infty \frac{x^{1/4}}{x^2+16}dx = \frac{\pi}{2\sqrt{2}}\left(\cos\left(\frac{\pi}{8}\right) - \sin\left(\frac{\pi}{8}\right)\right).$$

Question Two. We use the same contour as was in the last question of tutorial eight. We have $\gamma = \gamma_1 + \gamma_2 - \gamma_3$, where $\gamma_1(t) = t, 0 \le t \le R$, $\gamma_2(t) = Re^{it},\ 0 \le t \le \frac{2\pi}{5}$, $\gamma_3(t) = e^{\frac{2\pi i}{5}}t$, $0 \le t \le R$.

Let $f(z) = \frac{z}{z^5+1}$. The poles are at the solutions to $z^5 = e^{\pi i + 2k\pi i}$. The only pole inside the contour is at $z_1 = e^{\frac{\pi i}{5}}$. So that by the Residue Theorem

$$\begin{aligned}
&\int_0^R \frac{tdt}{1+t^5} + \int_0^{\frac{2\pi}{5}} \frac{iR^2e^{2it}}{R^5e^{5it}+1}dt - e^{\frac{4\pi i}{5}}\int_0^R \frac{tdt}{(e^{\frac{2\pi i}{5}}t)^5+1}\\
&= (1-e^{\frac{4\pi i}{5}})\int_0^R \frac{dt}{t^5+1} + \int_0^{\frac{2\pi}{5}} \frac{iR^2e^{2it}}{R^5e^{5it}+1}dt = 2\pi i\text{Res}\left(\frac{z}{1+z^5}, e^{\frac{\pi i}{5}}\right)\\
&= \frac{2\pi i}{5}e^{\frac{-3\pi i}{5}}.
\end{aligned}$$

Now

$$\begin{aligned}
\left|\int_0^{\frac{2\pi}{5}} \frac{iR^2e^{2it}}{R^5e^{5it}+1}dt\right| &\le \int_0^{\frac{2\pi}{5}} \left|\frac{iR^2e^{2it}}{R^5e^{5it}+1}\right| dt\\
&\le \int_0^{\frac{2\pi}{5}} \frac{R^2}{R^5-1}dt = \frac{R^2}{R^5-1}\frac{2\pi}{5} \to 0
\end{aligned}$$

as $R \to \infty$. So taking $R \to \infty$ we have

$$(1 - e^{\frac{4\pi i}{5}}) \int_0^\infty \frac{tdt}{t^5+1} = \frac{2\pi i}{5} e^{-\frac{3\pi i}{5}},$$

or

$$\begin{aligned}\int_0^\infty \frac{tdt}{t^5+1} &= \frac{2\pi i}{5} \frac{e^{-\frac{3\pi i}{5}}}{(1-e^{\frac{4\pi i}{5}})} \\ &= \frac{2\pi i}{5} \frac{e^{-\pi i}}{(e^{\frac{-2\pi i}{5}} - e^{\frac{2\pi i}{5}})} \\ &= \frac{\pi}{5\sin\left(\frac{2\pi}{5}\right)}.\end{aligned}$$

Question Three.
(a). We have $\int_0^{2\pi} f(e^{i\theta})d\theta$ for an analytic function f. Let $z = e^{i\theta}$ and C be the unit circle. Then we have

$$\begin{aligned}\int_0^{2\pi} f(e^{i\theta})d\theta &= \int_C \frac{f(z)}{iz}dz \\ &= 2\pi \frac{1}{2\pi i}\int_C \frac{f(z)}{z-0}dz = 2\pi f(0)\end{aligned}$$

by the Cauchy integral formula.
(a). We have $\int_0^{2\pi} f(e^{i\theta})\cos\theta d\theta$. As in the previous question we have

$$\begin{aligned}\int_0^{2\pi} f(e^{i\theta})\cos\theta d\theta &= \int_C f(z)\frac{1}{2}\left(z+\frac{1}{z}\right)\frac{dz}{iz} \\ &= \frac{1}{2i}\int_C f(z)dz + \frac{1}{2i}\int_C \frac{f(z)}{z^2}dz \\ &= \pi\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-0)^2}dz = \pi f'(0).\end{aligned}$$

The first integral on the second line is zero by Cauchy's Theorem.

Question Four.
(a)
We let $f(z) = \frac{1}{(z^2+1)^2}$ There are two poles of order 2 at $\pm i$. So we have

$$\begin{aligned}\text{Res}(\pi\cot(\pi z)f(z), i) &= \lim_{z\to i}\frac{d}{dz}\left(\frac{(z-i)^2\pi\cot(\pi z)}{(z-i)^2(z+i)^2}\right) \\ &= \frac{1}{4}(\pi^2 - \pi^2\coth^2\pi - \pi\coth\pi),\end{aligned}$$

$$\begin{aligned}\text{Res}(\pi\cot(\pi z)f(z), -i) &= \lim_{z\to -i}\frac{d}{dz}\left(\frac{(z+i)^2\pi\cot(\pi z)}{(z-i)^2(z+i)^2}\right) \\ &= \frac{1}{4}(\pi^2 - \pi^2\coth^2\pi - \pi\coth\pi).\end{aligned}$$

Since

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2+1)^2} = -\left(\mathrm{Res}(\pi\cot(\pi z)f(z), i) + \mathrm{Res}(\pi\cot(\pi z)f(z), -i)\right)$$
$$= \frac{1}{2}(\pi\coth\pi + \pi^2 \mathrm{cosech}^2\pi).$$

(b)

We take $f(z) = \frac{1}{z^4+4}$ which has four simple poles. These are at $z_1 = \sqrt{2}e^{i\frac{\pi}{4}}$, $z_2 = \sqrt{2}e^{-i\frac{\pi}{4}}$, $z_3 = \sqrt{2}e^{3i\frac{\pi}{4}}$, $z_4 = \sqrt{2}e^{-3i\frac{\pi}{4}}$. Then

$$\mathrm{Res}(f(z), z_1) = \lim_{z\to z_1} \frac{(z-z_1)\pi\cot(\pi z)}{z^4+4} = \frac{1}{16}(i-1)\pi\coth\pi,$$

$$\mathrm{Res}(f(z), z_2) = \lim_{z\to z_2} \frac{(z-z_2)\pi\cot(\pi z)}{z^4+4} = \frac{1}{16}(-i-1)\pi\coth\pi,$$

$$\mathrm{Res}(f(z), z_3) = \lim_{z\to z_3} \frac{(z-z_3)\pi\cot(\pi z)}{z^4+4} = \frac{1}{16}(-i-1)\pi\coth\pi,$$

$$\mathrm{Res}(f(z), z_4) = \lim_{z\to z_4} \frac{(z-z_4)\pi\cot(\pi z)}{z^4+4} = \frac{1}{16}(i-1)\pi\coth\pi.$$

This gives

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^4+4} = -\sum_{k=1}^{4} \mathrm{Res}(f(z), z_k) = \frac{\pi}{4}\coth\pi.$$

Question Five.

(a) We have the Laplace transform $F(s) = \dfrac{s^2-1}{(s^2+4)(s^2+9)}$. This has poles at $\pm 2i$ and $\pm 3i$. Then we require

$$\mathrm{Res}(F(s)e^{st}, 2i) = \frac{1}{4}ie^{2it},$$

$$\mathrm{Res}(F(s)e^{st}, -2i) = -\frac{1}{4}ie^{-2it},$$

$$\mathrm{Res}(F(s)e^{st}, 3i) = -\frac{1}{3}ie^{3it},$$

$$\mathrm{Res}(F(s)e^{st}, -3i) = \frac{1}{3}ie^{-3it},$$

If $f(t)$ is the inverse Laplace transform of $F(s)$ then

$$f(t) = \frac{1}{4}ie^{2it} - \frac{1}{4}ie^{-2it} - \frac{1}{3}ie^{3it} + \frac{1}{3}ie^{-3it}$$
$$= \frac{2}{3}\sin(3t) - \frac{1}{2}\sin(2t).$$

(b)

We have $F(s) = \dfrac{1}{s(e^s+1)}$. This has infinitely many poles. There are simple poles at $s = 0$ and $s = \pm\pi i, \pm 3\pi i$ .... Now

$$\mathrm{Res}(F(s)e^{st}, s = 0) = \frac{1}{2},$$

$$\mathrm{Res}(F(s)e^{st}, s = \pi i) = \frac{i}{\pi}e^{i\pi t},$$

$$\mathrm{Res}(F(s)e^{st}, s = -\pi i) = -\frac{i}{\pi}e^{-i\pi t},$$

$$\mathrm{Res}(F(s)e^{st}, s = 3\pi i) = \frac{i}{3\pi}e^{3i\pi t},$$

$$\mathrm{Res}(F(s)e^{st}, s = -3\pi i) = -\frac{i}{-3\pi}e^{3i\pi t},$$

etc. Thus the inverse Laplace transform is

$$f(t) = \frac{1}{2} - 2\sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi t)}{\pi(2n+1)}.$$