Complex Analysis Tutorial Five.

Question One.

To evaluate the line integral $\oint_C (\sin^3 x + 3y)\, dx + \left(x^2 - \frac{1}{\ln(y+3)}\right) dy$, we use Green's Theorem.

$$\frac{\partial P}{\partial y} = 3,$$
$$\frac{\partial Q}{\partial x} = 2x.$$

So

$$\oint_C (\sin^3 x + 3y)\, dx + \left(x^2 - \frac{1}{\ln(y+3)}\right) dy = \int\int_D (2x-3) dxdy$$
$$= \int\int_D 2x dxdy - 3\int\int dxdy.$$

D is the interior of a circle of radius 2 so it has area 4π. Hence

$$3\int\int dxdy = 12\pi.$$

Now C has equation $(x+2)^2 + (y-2)^2 = 4$. Put $x+2 = r\cos\theta$ and $y - 2 = r\sin\theta$, to get

$$\begin{aligned}
\int\int_D 2x dxdy &= 2\int_0^{2\pi}\int_0^2 (r\cos\theta - 2) r dr d\theta \\
&= 2\int_0^{2\pi}\left[\frac{r^3}{3}\cos\theta - r^2\right]_0^2 d\theta \\
&= 2\int_0^{2\pi} (\frac{8}{3}\cos\theta - 4) d\theta \\
&= 2\left[\frac{8}{3}\sin\theta - 4\theta\right]_0^{2\pi} = -16\pi.
\end{aligned}$$

Combining this gives

$$\oint_C (\sin^3 x + 3y)\, dx + \left(x^2 - \frac{1}{\ln(y+3)}\right) dy = \int\int_D (2x-3) dxdy$$
$$= -16\pi - 12\pi = -28\pi.$$

Question Two

We have the line integral around a triangle C $\int_C 2y^4 dx + 4xy^3 dy$. We have $P_y = 8y^3$ and $Q_x = 4y^3$. The region D enclosed by C is

$$D = \{(x,y) : 0 \le x \le 1,\ 3x \le y \le 3\}.$$

By the Theorem of Green we can say that

$$\begin{aligned}\int_C 2y^4dx + 4xy^3dy &= \int_0^1 \int_{3x}^3 (4y^3 - 8y^3)dydx \\ &= -\int_0^1 4y^3dydx \\ &= -\int_0^1 [y^4]_{3x}^3 dx \\ &= \int_0^1 [81 - 81x^4]dx = \left[81x - \frac{81}{5}x^5\right]_0^1 = -\frac{324}{5}.\end{aligned}$$

Question Three

We do not have differentiability here, so we cannot use the fundamental theorem. However the straight line is given by $\gamma(t) = a + bt$. If we take $t \in [0, 1]$ then $\gamma(0) = a = 1$ and $\gamma(1) = 1 + b = 2 + i$ so $b = i + 1$. Thus $\gamma(t) = 1 + (1 + i)t$ and $\gamma'(t) = (1 + i)$. Whence

$$\begin{aligned}\int_\gamma |z|^2dz &= \int_0^1 |1 + (1 + i)t|^2(1 + i)dt \\ &= \int_0^1 ((1 + t)^2 + t^2)(1 + i)dt \\ &= (1 + i)\int_0^1 (1 + 2t + 2t^2)dt \\ &= (1 + i)\left[t + t^2 + \frac{2}{3}t^3\right]_0^1 = \frac{8}{3}(1 + i).\end{aligned}$$

Question Four.

Put $\gamma(t) = 3 + i + 3e^{it}$. Then $\gamma'(t) = 3ie^{it}$. So that

$$\begin{aligned}\int_C \frac{1}{z - 3 - i}dz &= \int_0^{2\pi} \frac{3ie^{it}}{3 + i + 3e^{it} - 3 - i}dt \\ &= \int_0^{2\pi} idt = 2\pi i.\end{aligned}$$

Note that this is the same result we get integrating $1/z$ around the unit circle. Where the singularity is does not matter.

Question Five

If the circle does not contain zero then we can easily see that by the Fundamental Theorem of contour integration $\int_C \frac{1}{z^2}dz = 0$ since the function has an antiderivative throughout its domain. If the contour does contain zero we can assume without loss of generality that the circle is centred at zero. Then let $\gamma(t) = Re^{it},\ t \in [0, 2\pi)$. So that

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^{2\pi} \frac{1}{R^2 e^{2it}} iRe^{it} dt \\ &= \frac{i}{R} \int_0^{2\pi} e^{-it} dt \\ &= -\frac{1}{R}[e^{-it}]_0^{2pi} = -\frac{1}{R}[1-1] = 0.\end{aligned}$$

Question Six

We have the double integral $I = \int_0^1 \int_{2y}^2 e^{x^2} dxdy$. So the region is $0 \le y \le 1$ and $2y \le x \le 2$. So the same region is bounded by $0 \le x \le 2$ $0 \le y \le x/2$. Hence reversing the order of integration gives

$$\begin{aligned}I &= \int_0^2 \int_0^{\frac{x}{2}} e^{x^2} dydx \\ &= \int_0^2 \left[ye^{x^2}\right]_0^x dx \\ &= \int_0^2 \frac{x}{2} e^{x^2} dx \\ &= \frac{1}{4}[e^{x^2}]_0^2 = \frac{1}{4}(e^4 - 1).\end{aligned}$$

Question Seven.

In polar coordinates $\dfrac{1}{x^2+y^2} = \dfrac{1}{r^2}$. The region $a^2 \le x^2 + y^2 \le b^2$ is $a \le r \le b$ and $0 \le \theta \le 2\pi$ Hence

$$\begin{aligned}\int\int_{a^2 \le x^2+y^2 \le b^2} \frac{1}{x^2+y^2} dA &= \int_0^{2\pi} \int_a^b \frac{1}{r^2} r dr d\theta \\ &= \int_0^{2\pi} [\ln r]_a^b d\theta = \int_0^{2\pi} \ln(b/a) d\theta \\ &= 2\pi \ln(b/a).\end{aligned}$$

Question Eight.

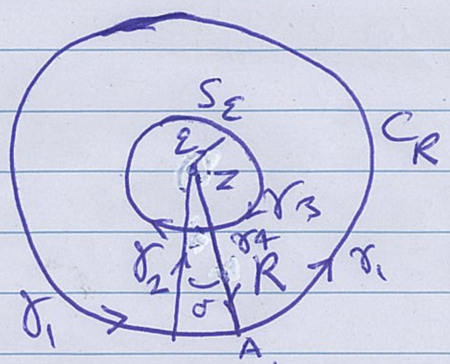
The Gaussian integral can be done in many ways. The easiest is to use a double integral and the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-y^2} dy$ so that

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} r e^{-r^2} dr d\theta \\
&= \int_{0}^{2\pi} \left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} d\theta \\
&= \int_{0}^{2\pi} \frac{1}{2} d\theta = \pi.
\end{aligned}
$$

Thus $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Note that we converted to polar coordinates to evaluate the double integral.

Q9 Tute 5

Suppose z is at the centre of D, though this is not essentially. We require C_R to be contained in D.



$$F(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$$

since $\lim_{\zeta \to z} F(\zeta) = f'(z)$

F is continuous in D

Consider contour γ that starts at A moves around C_R moves along γ_2, then clockwise around S_ε, then along γ_4 back to A.

By Cauchy's theorem

$$\int_\gamma F = 0$$

since γ does not contain z. Now as $\delta \to 0$

$\int_{\gamma_2} F \to -\int_{\gamma_4} F$ since the contours move in opposite directions

and $\gamma_1 \to C_R$, $\gamma_3 \to S_\varepsilon$

So $\int_{\gamma_1} F + \int_{\gamma_2} F + \int_{\gamma_3} F + \int_{\gamma_4} F = 0$ (*)

and as $\delta \to 0$ $\int_{\gamma_3} F \to -\int_{S_\varepsilon} F$

since γ_3 moves in opposite direction to S_ε.

Thus (*) $\to \int_{C_R} F + \int_{\gamma_2} F - \int_{\gamma_2} F - \int_{S_\varepsilon} F = 0$

or $\int_{C_R} F = \int_{S_\varepsilon} F$