Complex Analysis Tutorial Four Solutions

Question One.

We have $f(x+iy) = u(x,y) + iv(x,y)$ and we know $u(x,y) = 2x+y$. If f is entire, then u and v satisfy the CR equations. So $u_x = 2 = v_y$ and $u_y = 1 = -v_x$. From the first equation $v = 2y + h(x)$, h is an unknown function of x. Then $v_x = h'(x) = -1$. Thus $h(x) = -x + c$. Hence $v(x,y) = 2y - x + c$.

Thus $f(z) = f(x+iy) = 2x + y + i(2y - x + c) = 2(x+iy) - i(x+iy) + ic = (2-i)z + ic$.

Question Two.

By Euler's formula $\cos z = \cos(x+iy) = \frac{e^{i(x+iy)}+e^{-i(x+iy)}}{2}$. Take $x = 0$. Then $\cos(iy) = \cosh y$ which is unbounded.

Question Three.

We have

$$f(z) = e^{iz^2} = e^{i(x^2-y^2)-2xy} = e^{-2xy}(\cos(x^2-y^2) + i\sin(x^2-y^2)).$$

Then $u(x,y) = e^{-2xy}\cos(x^2-y^2)$. Differentiating gives

$$\begin{aligned}
u_x &= -2e^{-2xy}\left(x\sin\left(x^2-y^2\right) + y\cos\left(x^2-y^2\right)\right)\\
u_{xx} &= e^{-2xy}\left(2(4xy-1)\sin\left(x^2-y^2\right) - 4\left(x^2-y^2\right)\cos\left(x^2-y^2\right)\right)\\
u_y &= e^{-2xy}\left(2y\sin\left(x^2-y^2\right) - 2x\cos\left(x^2-y^2\right)\right),\\
u_{yy} &= e^{-2xy}\left(2(1-4xy)\sin\left(x^2-y^2\right) + 4\left(x^2-y^2\right)\cos\left(x^2-y^2\right)\right).
\end{aligned}$$

Clearly $u_{xx} + u_{yy} = 0$.

Question Four.

We have $f(z) = ze^{iz} = (x+iy)e^{ix-y} = e^{-y}(x+iy)(\cos x + i\sin x)$. We expand to obtain

$$f(x+i) = e^{-y}(x\cos(x) - y\sin(x)) + ie^{-y}(y\cos(x) + x\sin(x)).$$

Then $u_x = e^{-y}(-y\cos(x) - x\sin(x) + \cos(x)) = v_y$ and this holds for all x, y. Similarly $u_y = -e^{-y}\sin(x) - e^{-y}(x\cos(x) - y\sin(x)) = -v_x$. Again this holds for all x, y. So the CR equations are satisfied everywhere. Thus f is entire.

We also have $u_{xx} = e^{-y}((y-2)\sin(x) - x\cos(x)) = -u_{yy}$ so u is harmonic. Similarly $v_{xx} = -e^{-y}((y-2)\cos(x) + x\sin(x)) = -v_{yy}$.

For $f(z) = z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$ we have $u(x,y) = x^3 - 3xy^2, v = 3x^2y - y^3$ and clearly $u_x = 3x^2 - 3y^2 = v_y$ and $u_y = -6xy = -v_x$. These hold for all x, y so f is entire. One also has $u_{xx} = 6x = -v_{yy}$. So $u_{xx} + v_{yy} = 0$. $v_{xx} = 6y = -v_{yy}$. Thus $v_{xx} + v_{yy} = 0$ and u and v are harmonic.

Question Five.

We have $f(z) = 2z + z^2$ and $\gamma(t) = t + it^2$, $t \in [0,1]$. Then

$$
\begin{aligned}
\int_\gamma f(z)dz &= \int_0^1 f(\gamma(t))\gamma'(t)dt \\
&= \int_0^1 (2(t+it^2)+(t+it)^2)(2it+1)dt \\
&= \int_0^1 (-t^4+2it^3+(1+2i)t^2+2t)(2it+1)dt \\
&= \int_0^1 (2t+(1+6I)t^2-(4-4I)t^3-5t^4-2It^5)dt \\
&= \left[-\frac{it^6}{3}-t^5-(1-i)t^4+\left(\frac{1}{3}+2i\right)t^3+t^2\right]_0^1 \\
&= -\frac{2}{3}+\frac{8i}{3}.
\end{aligned}
$$

Question Six.

We have $f(z)=z^3$ and $\gamma(t)=e^{it}$, $t\in[0,2\pi)$. Then

$$
\begin{aligned}
\int_\gamma f(z)dz &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt \\
&= \int_0^{2\pi} e^{3it}ie^{it}dt \\
&= i\int_0^{2\pi} e^{4it}dt = \frac{1}{4}[e^{4it}]_0^{2\pi} = \frac{1}{4}(1-1) = 0.
\end{aligned}
$$

Question Seven.

We have

$$
\begin{aligned}
\int\int_D f dA &= \int_0^1 \int_{x^3}^{x^2} (x^4+y^2)dydx \\
&= \int_0^1 \left[x^4y+\frac{1}{3}y^3\right]_{x^3}^{x^2} dx \\
&= \int_0^1 \left(-\frac{x^9}{3}+\frac{x^6}{3}+\left(x^2-x^3\right)x^4\right)dx \\
&= \left[\frac{1}{3}\left(-\frac{x^{10}}{10}-\frac{3x^8}{8}+\frac{4x^7}{7}\right)\right]_0^1 \\
&= \frac{9}{280}.
\end{aligned}
$$

Now we reverse the order of integration. $y = x^3$ gives $x = y^{1/3}$ and $y = x^2$ gives $x = y^{1/2}$. The other limits are 0 and 1. So We have

$$
\begin{aligned}
\int\int_D f dA &= \int_0^1 \int_{y^{1/2}}^{y^{1/3}} (x^4 + y^2) dx dy \\
&= \int_0^1 \left[\frac{1}{5} x^4 + xy^2 \right]_{y^{1/2}}^{y^{1/3}} \\
&= \int_0^1 \left(-\frac{y^{5/2}}{5} + \frac{y^{5/3}}{5} + \left(\sqrt[3]{y} - \sqrt{y}\right) y^2 \right) dy \\
&= \left[\frac{1}{5} \left(-\frac{12y^{7/2}}{7} + \frac{3y^{10/3}}{2} + \frac{3y^{8/3}}{8} \right) \right]_0^1 \\
&= \frac{9}{280}.
\end{aligned}
$$