Complex Analysis Tutorial Six Solutions.

Question One.

We let

$$\oint_C \left(\ln\left(\frac{1}{\sqrt{x}}\right) + 3y^2\right) dx + \left(x - \frac{\cos y}{y^2+1}\right) dy = \oint_C Pdx + Qdy.$$

Then $Q_x = 1$ and $P_y = 6y$. C is the circle $(x-1)^1 + (y-2)^2 = 25$. This has radius 5. We let $x = 1 + r\cos\theta, y = 2 + r\sin\theta$. So $0 \leq r \leq 5$ and $0 \leq \theta \leq 2\pi$. Let D be the interior of C. By Green's Theorem

$$\begin{aligned}
\oint_C Pdx + Qdy &= \int\int_D (Q_x - P_y)dA \\
&= \int_0^{2\pi}\int_0^5 (1 - 6(2 + r\sin\theta))rdrd\theta \\
&= -\int_0^{2\pi}\int_0^5 (11r + 6r^2\sin\theta)drd\theta \\
&= -\int_0^{2\pi}\left[\frac{11}{2}r^2 + 2r^3\sin\theta\right]_0^5 d\theta \\
&= -\int_0^{2\pi}\left(\frac{275}{2} + 250\sin\theta\right)d\theta \\
&= -\left[\frac{275}{2}\theta - 250\cos\theta\right]_0^{2\pi} = -275\pi.
\end{aligned}$$

Question Two.

We have $\gamma(t) = 2e^{it}, t \in [0, \pi/2]$. Then

$$\begin{aligned}
\int_C (z^3 + iz)dz &= \int_0^{\frac{\pi}{2}} ((2e^{it})^3 + 2ie^{it})2ie^{it}dt \\
&= 2i\int_0^{\frac{\pi}{2}} (8e^{4it} + 2ie^{2it})dt \\
&= \left[4e^{4it} + 2ie^{2it}\right]_0^{\frac{\pi}{2}} \\
&= 4e^{2\pi i} + 2ie^{\pi i} - 4 - 2i = -4i.
\end{aligned}$$

By the Fundamental Theorem of Contour Integration

$$\int_C (z^3 + iz)dz = \int_2^{2i} (z^3 + iz)dz = \left[\frac{1}{4}z^4 + \frac{1}{2}iz^2\right]_2^{2i} = -4i.$$

Question Three.

We have $f(z) = z^2 - 2z$ and since this is differentiable, the integral is independent of path. Hence

$$\int_i^{2i} (z^2 - 2z)dz = \left[\frac{1}{3}z^3 - z^2\right]_i^{2i} = 3 - \frac{7}{3}i.$$

Question Four.

We can set up the integral as a sum of integrals along four straight line contours. However by the Cauchy integral formula, with $f(z) = 1$

$$\int_C \frac{1}{z-i} dz = 2\pi i f(1) = 2\pi i.$$

Question Five.

Notice that the singularity of the integrand does not lie inside the curve. So the function is differentiable on and inside C. Thus by Cauchy't Theorem

$$\int_C \frac{\ln(z-i)}{z+i} dz = 0.$$

Question Six.

Green's Theorem says that

$$\oint P dx + Q dy = \int\int_D (Q_x - P_y) dA.$$

D is the interior of C. Let $P = -\frac{\partial u}{\partial y}$ and $Q = \frac{\partial u}{\partial x}$. Then

$$\oint_C \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) = \int\int_D \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dxdy.$$

If u is a harmonic function then

$$\oint_C \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) = 0.$$

This is important in the study of Laplace's equation.

Question Seven. We have

$$\nabla F . \nabla G = \frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y}.$$

Put $P = F\frac{\partial G}{\partial y}$ and $Q = -F\frac{\partial G}{\partial x}$. Then

$$\begin{aligned}
\oint_C P dx + Q dy &= \oint_C F G_y dx - F G_x dy \\
&= \int\int_R \left(\frac{\partial}{\partial x}(-F G_x) - \frac{\partial}{\partial y}(F G_y) \right) dxdy \\
&= \int\int_R (-F G_{xx} - F_x G_x - F_y G_y - F G_{yy}) dxdy \\
&= -\int\int_R (F \Delta G + \nabla F . \nabla G) dxdy.
\end{aligned}$$

Question Eight.

We write the integral as

$$\oint_C \frac{z^3-6}{2z-i}dz = \oint_C \frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i}dz.$$

Now put $f(z) = \frac{1}{2}z^3 - 3$. Then by the Cauchy integral formula

$$\begin{aligned}\oint_C \frac{z^3-6}{2z-i}dz &= 2\pi i f(i/2) \\ &= \frac{\pi}{8} - 6\pi i.\end{aligned}$$

Question Nine.

Let $f(z) = 2z^3$. $z_0 = 2$ is inside the curve C. So by the general Cauchy integral formula

$$\begin{aligned}\oint_C \frac{f(z)}{(z-2)^2}dz &= 2\pi i f'(z_0) \\ &= 2\pi i 6(2)^2 = 48\pi i.\end{aligned}$$

Question Ten.

Put $f(z) = e^z$. Then by the Cauchy integral formula

$$\int_C \frac{e^z}{z}dz = 2\pi i f(0) = 2\pi i.$$

Now put $\gamma(t) = e^{it}$ and let $t \in [0, 2\pi i)$. Then

$$\begin{aligned}\int_C \frac{e^z}{z}dz &= \int_0^{2\pi} \frac{e^{e^{it}}}{e^{it}} ie^{it}dt \\ &= i\int_0^{2\pi} e^{\cos t + i\sin t}dt \\ &= i\int_0^{2\pi} e^{\cos t}(\cos(\sin t) + i\sin(\sin t))dt = 2\pi i.\end{aligned}$$

Equating the real and imaginary parts we have

$$\int_0^{2\pi} e^{\cos t}(\cos(\sin t)dt = 2\pi$$

and

$$\int_0^{2\pi} e^{\cos t}(\sin(\sin t)dt = 0.$$

Question Eleven.

By elementary properties of integrals

$$\left|\int_C \frac{e^{iz}}{z^3+z^2}dz\right| \leq \int_C \left|\frac{e^{iz}}{z^3+z^2}\right| dz$$
$$= \int_C \frac{1}{|z^3+z^2|}dz$$

It is obvious that $|z^3 + z^2| \geq |z|^2$ for $|z+1| > 1$. Taking reciprocals we have

$$\frac{1}{|z^3+z^2|} \leq \frac{1}{|z|^2} \leq \frac{1}{R^2}.$$

So by the ML inequality we have

$$\left|\int_C \frac{e^{iz}}{z^3+z^2}dz\right| \leq \frac{2\pi R}{R^2} = \frac{2\pi}{R},$$

because the length of the contour is $2\pi R$.