Complex Analysis Tutorial Eight Solutions

Question One. We have

$$\begin{aligned} f(z) &= \frac{1}{(z-5)(z-3)} = \frac{1}{(z-3)((z-3)-2)} \\ &= \frac{1}{(z-3)^2\left(1-\frac{2}{z-3}\right)} \\ &= \frac{1}{(z-3)^2}\left(1+\frac{2}{z-3}+\left(\frac{2}{z-3}\right)^2+\cdots\right) \end{aligned}$$

This converges for $\left|\frac{2}{z-3}\right| < 1$. We can also write

$$\begin{aligned} f(z) &= \frac{1}{(z-5)(z-3)} = \frac{1}{(z-3)((z-3)-2)} \\ &= \frac{-1}{2(z-3)}\frac{1}{1-\frac{z-3}{2}} \\ &= \frac{-1}{2(z-3)}\left(1+\frac{z-3}{2}+\left(\frac{z-3}{2}\right)^2+\cdots\right), \end{aligned}$$

which converges for $|z-3| < 2$. The residue at $z = 3$ is clearly $-1/2$.

Question Two.

$$\begin{aligned} f(z) &= \frac{1}{(z-4)(z+1)} = \frac{1}{(z+1)((z+1)-5)} \\ &= \frac{-1}{5(z+1)}\frac{1}{1-\left(\frac{z+1}{5}\right)} \\ &= \frac{-1}{5(z+1)}\left(1+\left(\frac{z+1}{5}\right)+\left(\frac{z+1}{5}\right)^2+\cdots\right), \end{aligned}$$

for $|z+1| < 5$. The residue at -1 is $-1/5$. Now $z = -1$ is the only pole inside C, so

$$\int_C \frac{1}{(z-4)(z+1)}dz = 2\pi \times \frac{-1}{5} = \frac{-2\pi i}{5}.$$

Question Three. We use the substitution $z = e^{i\theta}$.
(a)

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} &= \int_C \frac{1}{\left(5-\frac{3}{2i}(z-1/z)\right)^2}\frac{dz}{iz} \\ &= \int_C \frac{4iz}{(3+10iz-3z^2)^2}dz. \end{aligned}$$

There are poles of order 2 at the roots of $3+10iz-3z^2 = 0$. The poles are $z = 3i$ and $i/3$. Clearly only the second is inside the unit circle.

$$\begin{aligned}\operatorname{Res}(f(z), i/3) &= \lim_{z\to i/3}\frac{d}{dz}\left(\frac{(z-i/3)^2 4iz}{(3+10iz-3z^2)^2}\right)\\ &= \lim_{z\to i/3}\frac{4(3-iz)}{9(z-3i)^3} = -\frac{5i}{64}.\end{aligned}$$

So by the Residue theorem

$$\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = 2\pi i\left(-\frac{5i}{64}\right) = \frac{5\pi}{32}.$$

(b)

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{3+\cos\theta-2\sin\theta} &= \int_C \frac{1}{3+\frac{1}{2}(z+1/2)+i(z-1/z)}\frac{dz}{iz}\\ &= \int_C \frac{2}{i}\frac{dz}{(1+2i)z^2+6z+1-2i}\end{aligned}$$

There are poles at $z_1 = -1+2i$ and $z_2 = -\frac{1}{5}+\frac{2i}{5}$. z_2 is inside the unit circle. The other pole is outside.

$$\operatorname{Res}(f(z), -\frac{1}{5}+\frac{2i}{5}) = \lim_{z\to -\frac{1}{5}+\frac{2i}{5}}\frac{2}{i}\frac{(z-(-\frac{1}{5}+\frac{2i}{5})}{(1+2i)z^2+6z+1-2i} = \lim_{z\to -\frac{1}{5}+\frac{2i}{5}}\frac{2}{i}\frac{1}{z-z_1} = -i/2.$$

Thus

$$\int_0^{2\pi}\frac{d\theta}{3+\cos\theta-2\sin\theta} = 2\pi i\,(-i/2) = \pi.$$

(c)

$$\begin{aligned}\int_0^{2\pi}\frac{\cos(3\theta)}{4-\cos\theta}d\theta &= \int_C \frac{1}{2}\left(z^3+\frac{1}{z^3}\right)\frac{1}{4-\frac{1}{2}(z+\frac{1}{z})}\frac{dz}{iz}\\ &= \int_C \frac{i(z^6+1)}{2z^3(2z^2-5z+2)}dz.\end{aligned}$$

Clearly there is a pole of order 3 at $z=0$. There are two other poles at $z=1/2$ and $z=2$. Thus we have two poles in the unit circle.

$$\operatorname{Res}(f(z), 1/2) = \lim_{z\to 1/2}\frac{i(z-1/2)(z^6+1)}{2z^3(2z^2-5z+2)} = \frac{-65i}{48}.$$

$$\operatorname{Res}(f(z), 0) = \lim_{z\to 0}\frac{1}{2}\frac{d^2}{dz^2}\left(\frac{iz^3(z^6+1)}{2z^3(2z^2-5z+2)}\right) = \frac{21i}{16}.$$

Do this in Mathematica. It is far too messy by hand. In practice most of these sorts of problems would be done on a computer in real life.

So

$$\int_0^{2\pi}\frac{\cos(3\theta)}{4-\cos\theta}d\theta = 2\pi i\left(\frac{21i}{16}+\frac{-65i}{48}\right) = \frac{\pi}{12}.$$

Question Four. We will do these in part 2.