Complex Analysis Tutorial One

Question One.

We have $f(x) = |x-1|$. From the definition of the derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Now we consider limits to the left and right. We want $x = 1$.

$$\begin{aligned}\lim_{h\to 0^+} \frac{|x+h-1|-|x-1|}{h}\Big|_{x=1} &= \lim_{h\to 0^+} \frac{|1+h-1|-|1-1|}{h} \\ &= \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1,\end{aligned}$$

since $h > 0$ implies $|h| = h$. Now we take the limit from the left.

$$\begin{aligned}\lim_{h\to 0^-} \frac{|x+h-1|-|x-1|}{h}\Big|_{x=1} &= \lim_{h\to 0^-} \frac{|1+h-1|-|1-1|}{h} \\ &= \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1,\end{aligned}$$

since $h < 0$ implies that $|h| = -1$. These two limits are different.

Thus f' is discontinuous at $x = 1$.

Question Two. In both these examples the limits are of the form $\frac{0}{0}$.

So we can apply L'Hôpital's rule.

(a)

$$\lim_{x\to 0} \frac{\tan x}{2x} = \lim_{x\to 0} \frac{\sec^2 x}{2} = \frac{1}{2}.$$

(b)

$$\lim_{x\to -2} \frac{x+2}{x^3+8} = \lim_{x\to -2} \frac{1}{3x^2} = \frac{1}{12}.$$

Question Three.

(a) $f(x,y) = x^2+y^2$. Then $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$.

(b) $f(x,y) = x^3y$. Then $\frac{\partial f}{\partial x} = 3x^2y$, $\frac{\partial f}{\partial y} = x^3$.

(c) $f(x,y) = x\ln(xy^2) = x\ln x + 2x\ln y$. So

$$\frac{\partial f}{\partial x} = \ln x + 1 + 2\ln y = \ln(xy^2)+1, \frac{\partial f}{\partial y} = \frac{2x}{y}.$$

(d) $f(x,y) = e^{-x^2-y^2}$, $\frac{\partial f}{\partial x} = -2xf(x,y)$, $\frac{\partial f}{\partial x} = -2yf(x,y)$.

(e) $f(x,y) = \sin(xy^2)$. $\frac{\partial f}{\partial x} = y^2\cos(xy^2)$, $\frac{\partial f}{\partial y} = 2xy\cos(xy^2)$.

Question Four.

The principal value of the argument of i is $\frac{\pi}{2}$. So that $i = e^{\frac{\pi}{2}i}$. Hence $1/i = e^{-\frac{\pi}{2}i}$.

Question Five.
(a) Let $z = x + iy$. Then $|z| = \sqrt{x^2 + y^2}$. So $|z| = 2$ is the set of points $\sqrt{x^2 + y^2} = 2$, or $x^2 + y^2 = 4$. This is a circle of radius 2 centered at the origin. See accompanying Mathematica file.
(b) $|z + 1| = |x + iy + 1| = \sqrt{(x + 1)^2 + y^2}$. Thus $|z + 1| = 1$ is the set of points $\sqrt{(x + 1)^2 + y^2} = 1$ which is the same as $(x + 1)^2 + y^2 = 1$. This is a circle of radius 1 centered at $(-1, 0)$. See Mathematica file.
(c) $|z - 2i| = |x + iy - 2i| = \sqrt{x^2 + (y - 2)^2}$. Hence $|z - 2i| \leq 1$ is the set of points $\sqrt{x^2 + (y - 2)^2} \leq 1$ or $x^2 + (y - 2)^2 \leq 1$ which is the interior of the circle of radius 1, centered at $(0, 2)$. See Mathematica file.
(d) $\mathrm{Arg}(z) = \frac{2\pi}{3}$ is the set of points of the form $z = re^{\frac{2\pi}{3}i}$. For any $r > 0$.
(e) $|z - 2| = \sqrt{(x - 2)^2 + y^2}$ So $|z - 2| > 1$ and $|z - 2| < 3$ is the set of points $1 < |z - 2| < 3$ or $1 < (x - 2)^2 + y^2 < 9$. This is the region between the circles centered at $(2, 0)$ of radii 1 and 3. See Mathematica file.

Question Six.
Euler's formula says $e^{iz} = \cos z + i \sin z$. Hence $e^{-iz} = \cos z - i \sin z$. So that $e^{iz} + e^{-iz} = \cos z + i \sin z + \cos z - i \sin z = 2 \cos z$. Further $e^{iz} - e^{-iz} = \cos z + i \sin z - (\cos z - i \sin z) = 2i \sin z$. Rearranging these gives the results.
Now

$$\begin{aligned}\cos^2 z + \sin^2 z &= \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 \\ &= \frac{e^{2iz} + 2 + e^{-2iz}}{4} - \frac{e^{2iz} - 2 + e^{-2iz}}{4} = \frac{4}{4} = 1.\end{aligned}$$

Question Seven.
We have to solve $z^3 = 1 + i$. We $1 + i$ write in polar form. $|1 + i| = \sqrt{1 + 1} = \sqrt{2}$. The number is in the first quadrant, so the principal value is $\frac{\pi}{4}$. Thus we want to solve $z^3 = e^{\frac{\pi}{4}i + 2k\pi i}$. Taking $k = 0$, we see that the first solution z_0 must satisfy $z_0^3 = \sqrt{2}e^{\frac{\pi}{4}i}$. Hence $z_0 = 2^{1/6}e^{\frac{\pi}{12}i}$. Next take $k = 1$. So the second solution satisfies $z_1^3 = \sqrt{2}e^{\frac{\pi}{4}i + 2\pi i} = \sqrt{2}e^{\frac{9\pi}{4}i}$. Hence $z_1 = 2^{1/6}e^{\frac{3\pi}{4}i}$. Finally take $k = -1$. Then the third solution satisfies $z_{-1}^3 = \sqrt{2}e^{\frac{\pi}{4}i - 2\pi i} = \sqrt{2}e^{\frac{-7\pi}{4}i}$. Hence $z_{-1} = 2^{1/6}e^{-\frac{7\pi}{12}i}$.