Complex Analysis Tutorial Two

Question One

We let $z = a + ib$ and suppose that $b \neq 0$. Now

$$\sin(z) = \sin(a + ib) = \sin a \cos(ib) + \sin(ib) \cos a$$
$$= \sin a \cosh b + i \sinh b \cos a$$

Now if $\sin z = 0$ the real and imaginary parts must both be zero. Hence $\sin a \cosh b = 0$. However $\cosh b = \frac{1}{2}(e^b + e^{-b}) \neq 0$ for any real b, since the exponential of a real number is positive. This means that $\sin a = 0$. Since a is real, then it is one of the real roots of $\sin x$. These are $a = n\pi$ where n is an integer.

Next we must also have $\sinh b \cos a = 0$. However $a = n\pi$. Hence $\cos(n\pi) \sinh b = (-1)^n \sinh b = 0$. This means that $\sinh b = frac12(e^b - e^{-b}) = 0$. This is only possible if $b = 0$. So if $\sin z = 0$, $z = n\pi$ and so all the zeroes are real.

We repeat this for $\cos z = \cos(a+ib) = \cos a \cos(ib) - \sin a \sin(ib) = \cos a \cosh b - i \sin a \sinh b$. We apply the same argument as before. $\cos a \cosh b = 0$ implies that $\cos a = 0$, so $a = (n + \frac{1}{2})\pi$ for n an integer. Since $\sin a \sinh b = \sin((n + \frac{1}{2})\pi) \sinh b = (-1)^n \sinh b = 0$ we must have $b = 0$. So that $z = (n + \frac{1}{2})\pi$ and all zeroes are real.

Question Two

We have the power series expansion $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Now let $a_n = \frac{z^n}{n!}$. By the ratio test, the series will converge absolutely if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$. Now

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{z^{n+1}}{(n+1)!} / (\frac{z^n}{n!}) \right| = \lim_{n\to\infty} \frac{|z| n!}{(n+1)!}$$
$$= \lim_{n\to\infty} \frac{|z|}{n+1} = |z| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

So the series converges uniformly for all $z \in \mathbb{C}$.

Next we have $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n+1}}{(2n+1)!}$. For the ratio test we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+2} z^{2n+3}}{(2n+3)!} / (\frac{(-1)^n z^{2n+1}}{(2n+1)!}) \right| = \lim_{n\to\infty} \frac{|z|^2 (2n+1)!}{(2n+3)!}$$
$$= \lim_{n\to\infty} \frac{|z|^2}{(2n+2)(2n+3)} = |z|^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)}$$
$$= 0 < 1.$$

Finally $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n}}{(2n)!}$. Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+2} z^{2n+2}}{(2n+2)!} / (\frac{(-1)^n z^{2n}}{(2n)!}) \right| = \lim_{n\to\infty} \frac{|z|^2 (2n)!}{(2n+2)!}$$
$$= \lim_{n\to\infty} \frac{|z|^2 \quad 1}{(2n+1)(2n+2)} = |z|^2 \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)}$$
$$= 0 < 1.$$

Question Three

We have $\tanh z = \frac{\sinh z}{\cosh z} = y$. So that $\frac{e^z - e^{-z}}{e^z + e^{-z}} = y$. This can be rewritten $\frac{e^{2z}-1}{e^{2z}+1} = y$. Or $(e^{2z} - 1) = y(e^{2z} + 1)$. Rearranging we have

$$e^{2z}(1 - y) = y + 1.$$

Hence

$$e^{2z} = \frac{y+1}{y-1}.$$

We take logs to get

$$z = \tanh^{-1}(y) = \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right).$$

Question Four

Now $\cosh z = \frac{1}{2}(e^z + e^{-z}) = y$. Multiply both sides by $2e^z$ to produce $e^{2z} + 1 = 2e^z y$. If we put $u = e^z$ we have the quadratic equation $u^2 - 2uy + 1 = 0$. Then $u = \frac{2y \pm \sqrt{4y^2 - 4}}{2} = y \pm \sqrt{y^2 - 1}$. Note $y = \cosh z \geq 1$. We take the positive square root, since $\cosh z$ is positive when z is real. Now take the natural log to produce

$$z = \cosh^{-1}(y) = \ln(y + \sqrt{y^2 - 1}).$$

Question Five

We want to find the roots of $z^6 = 1$. Lets call them $z_1, ..., z_6$. We write $z^6 = e^{2k\pi i}$. Then we use the fact that the roots appear in complex conjugate pairs. Take $k = 0$, then the first root satisfies $z_1^6 = 1$, so $z_1 = 1$. We also see that $z_2 = -1$ is also a root. Next take $k = 1$. The third root satisfies $z_3^6 = e^{2k\pi i}$, or

$$z_3 = e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{3}} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

. We then immediately see that the fourth root is

$$z_4 = e^{\frac{-\pi i}{3}} = \cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

. Taking $k = 2$ we get the fifth root $z_5^6 = e^{4\pi i}$, or $z_5 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$. and the final root is $z_6 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$.

Question Six

We write $z = (2 - 2i) = |z|e^{iargz}$. Now $|z| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$. $argz = -\frac{\pi i}{4} + 2k\pi i$. Proceeding as in question five we have four roots: $8^{1/8}e^{-\frac{\pi i}{16}}, (k = 0)$, $8^{1/8}e^{\frac{7\pi i}{16}}, (k = 1)$, $8^{1/8}e^{\frac{-9\pi i}{16}}, (k = -1)$, $8^{1/8}e^{\frac{15\pi i}{16}}, (k = 2)$.

Question Seven

We have from the definitions of the hyperbolic functions

$$\begin{aligned}\cosh^2 z - \sinh^2 z &= \left(\frac{1}{2}(e^z + e^{-z})\right)^2 - \left(\frac{1}{2}(e^z - e^{-z})\right)^2 \\ &= \frac{1}{4}(e^{2z} + 2 + e^{-2z} - (e^{2z} - 2 + e^{-2z}) \\ &= \frac{4}{4} = 1.\end{aligned}$$

Question Eight

This uses the fairly obvious fact that $Arg(e^x e^{iy}) = y$. We let $z = x + iy$, note that we are taking the principal value and write

$$\begin{aligned}Log_e(e^z) &= \ln|e^z| + iArg(e^x e^{iy}) \\ &= \ln(e^x) + iy \\ &= x + iy = z.\end{aligned}$$

We used the fact that $|e^{iy}| = 1$. So we have shown that the principal value of the logarithm is the inverse of the exponential function.

Question Nine

We write $i = e^{\frac{\pi i}{2} + 2k\pi i}$, $k \in \mathbb{Z}$. Then

$$i^{i/2} = (e^{\frac{\pi i}{2} + 2k\pi i})^{i/2} = e^{-\frac{\pi}{4} - k\pi}.$$

Question Ten

We want to calculate $\sum_{k=1}^{n} \sin(k\theta)$. We use Euler's formula and the sum of a geometric progression. Recall that

$$ar + ar^2 + \cdots + ar^n = \frac{ar\,(r^n - 1)}{r - 1}$$

Now let $r = e^{i\theta}$ and $a = 1$. Then

$$\begin{aligned}\sum_{k=1}^{n} \sin(k\theta) &= Im \sum_{k=1}^{n} e^{ik\theta} \\ &= \frac{e^{i\theta}(e^{in\theta} - 1)}{e^{i\theta} - 1}.\end{aligned}$$

Now

$$\begin{aligned}
\frac{e^{i\theta}(e^{in\theta}-1)}{e^{i\theta}-1} &= \frac{e^{i\theta}e^{in\frac{\theta}{2}}(e^{in\frac{\theta}{2}}-e^{-in\frac{\theta}{2}})}{e^{i\frac{\theta}{2}}(e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}})} \\
&= \frac{e^{i\frac{\theta}{2}}e^{in\frac{\theta}{2}}(2i\sin\left(\frac{n\theta}{2}\right))}{2i\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{e^{i\frac{1}{2}(n+1)\theta}\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \\
&= \left(\cos\left(\frac{1}{2}(n+1)\theta\right)+i\sin\left(\frac{1}{2}(n+1)\theta\right)\right)\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}
\end{aligned}$$

Taking the imaginary part we obtain the formula

$$\sum_{k=1}^{n}\sin(k\theta) = \sin\left(\frac{1}{2}(n+1)\theta\right)\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$