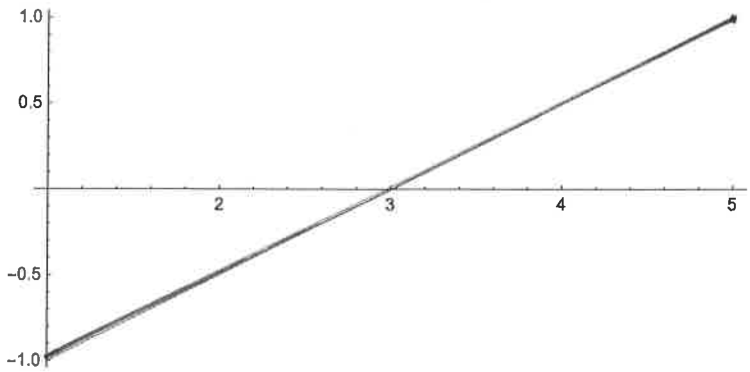
```
ParametricPlot[{2 t + 1, t - 1}, {t, 0, 2}]
```

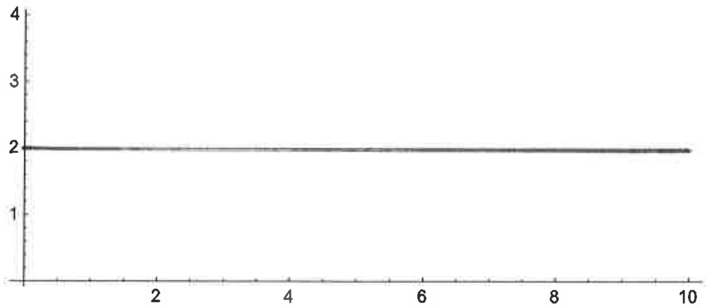
1 a)



$\gamma(t) = 2t+1 + i(t-1)$

```
ParametricPlot[{t, 2}, {t, 0, 10}]
```

b)

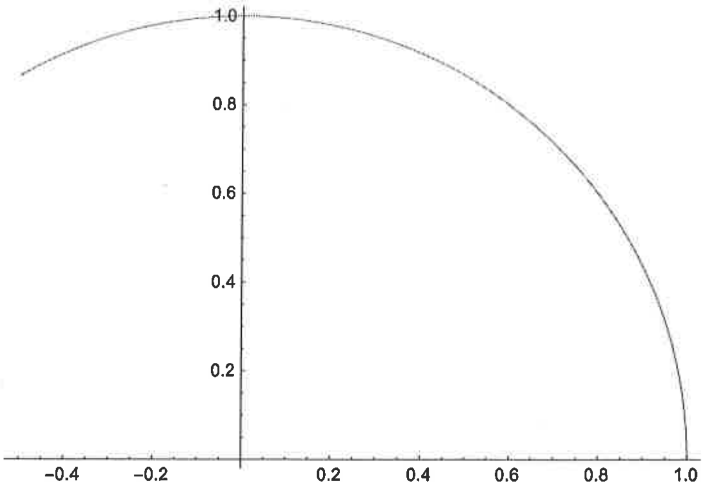


$\gamma(t) = t + 2i$

$t \geq 0$

```
ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2 π / 3}]
```

c)

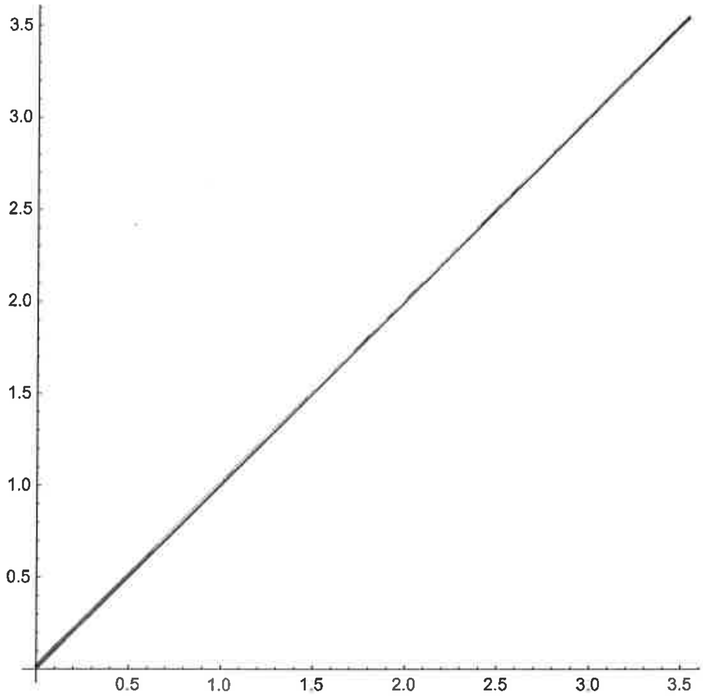


$\gamma(t) = \cos t + i \sin t$

$t \in [0, \frac{2\pi}{3}]$

d)

```
ParametricPlot[{t Cos[π / 4], t Sin[π / 4]}, {t, 0, 5}]
```

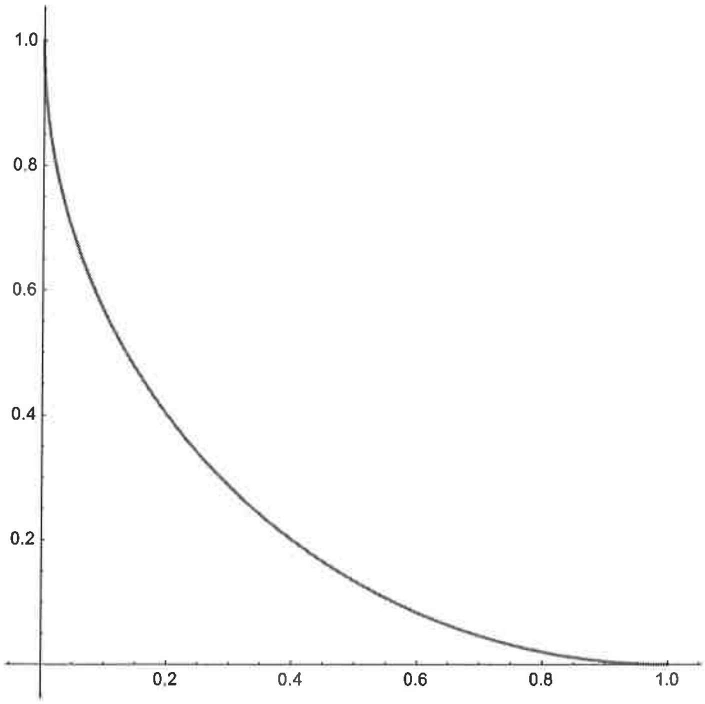


$$\gamma(t) = te^{i\pi/4} = t\cos(\tfrac{\pi}{4}) + it\sin(\tfrac{\pi}{4})$$

3(b)

```
In[3]:= ParametricPlot[{1 + Cos[t], 1 - Sin[t]}, {t, π / 2, π}]
```

Out[3]=



$$\gamma(t) = 1 + i + e^{-it} = 1 + \cos t + i(1 - \sin t)$$

$x(t) = 1 + \cos t$

$y(t) = 1 - \sin t$

and

$(x(t) - 1)^2 + (y(t) - 1)^2 = 1.$

3(a) $\gamma(t) = x(t) + iy(t)$ $\quad (x(t) - 3)^2 + y(t)^2 = 4$

$x(t) - 3 = 2\cos t, \; y(t) = 2\sin t$

$\therefore x(t) = 3 + 2\cos t \quad y(t) = 2\sin t$

$$\therefore \gamma(t) = 3 + 2\cos t + i2\sin t = 3 + 2e^{it}, \quad t \in [0, 2\pi)$$

2 (a) $\gamma(t) = (4+i)t, \quad t \in [0,1]$

(b) $\gamma(t) = a + bt \quad \gamma(0) = a = i$

$\gamma(1) = i + b = 3 + i \quad \therefore b = 3$

$\therefore \gamma(t) = i + 3t, \quad t \in [0,1]$

(c) $\gamma(t) = a + bt \quad \gamma(0) = 1 - 4i = a$

$\gamma(1) = 1 - 4i + b = 1 + 4i$

$\therefore b = 8i$

$\gamma(t) = 1 - 4i + 8it$

$= 1 + 4i(2t-1), \quad t \in [0,1]$

Complex Analysis Tutorial Three. Solutions part 2.

Question Four.

We know that $f'(z_0)$ exists. So that

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Now because this limit exists we can write

$$\begin{aligned}
\lim_{z\to z_0} (f(z) - f(z_0)) &= \lim_{z\to z_0} (z - z_0)\left(\frac{f(z) - f(z_0)}{z - z_0}\right) \\
&= \lim_{z\to z_0} (z - z_0) \lim_{z\to z_0} (z - z_0)\left(\frac{f(z) - f(z_0)}{z - z_0}\right) \\
&= \lim_{z\to z_0} (z - z_0) f'(z_0) = 0.
\end{aligned}$$

Hence $\lim_{z\to z_0} f(z) = f(z_0)$, so that f is continuous at z_0.

Question Five.

The function $f(z) = (\bar{z})^2 = (x - iy)^2 = x^2 - y^2 - 2ixy$. So $u(x, y) = x^2 - y^2$ $v(x, y) = -2xy$. Now $u_x = 2x, u_y = -2y$ and $v_x = -2y, v_y = -2x$. So unless $x, y = 0$ the CR equations are not satisfied. So f is not differentiable anywhere except at zero.

Question Six.

We have the function $f(x + iy) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$. For f to be differentiable, the CR equations must be satisfied. Now $u_x = 2x + ay$, $u_y = ax + 2by$, $v_x = 2cx + dy$ and $v_y = dx + 2y$. We require $u_x = v_y$ and $u_y = -v_x$. That is $2x + ay = dx + 2y$. This means $d = 2$ and $a = 2$. We also require $ax + 2by = -2cx - dy = -2cx - 2y$. So $b = -\frac{1}{2}$ and $-2c = a = 2$, or $c = -1$.

Question Seven.

By Euler's formula

$$\begin{aligned}
e^{-z^2} &= e^{-(x+iy)^2} = e^{-(x^2-y^2+2ixy)} \\
&= e^{-x^2+y^2-2ixy} = e^{-x^2+y^2}(\cos(2xy) - i\sin(2xy)).
\end{aligned}$$

From which we have $u(x, y) = e^{-x^2+y^2}\cos(2xy)$, $v(x, y) = -e^{-x^2+y^2}\sin(2xy)$. We see that

$$\begin{aligned}
u_x &= -2xe^{-x^2+y^2}\cos(2xy) - 2ye^{-x^2+y^2}\sin(2xy), \\
u_y &= 2ye^{-x^2+y^2}\cos(2xy) - 2xe^{-x^2+y^2}\sin(2xy), \\
v_x &= 2xe^{-x^2+y^2}\sin(2xy) - 2ye^{-x^2+y^2}\cos(2xy), \\
v_y &= -2ye^{-x^2+y^2}\sin(2xy) - 2xe^{-x^2+y^2}\cos(2xy).
\end{aligned}$$

Obviously $u_x = v_y$ and $u_y = -v_x$ for every x, y. So it is clear that for every x, y the CR equations are satisfied. Thus f is differentiable everywhere, so that it is entire.

Question Eight.

If $f(x+iy) = x^3 - 3xy^2 + iv(x,y)$ is differentiable, then $u_x = 3x^2 - 3y^2 = v_y$ and $u_y = -6xy = -v_x$. So integrating v with respect to x gives $v = 3x^2y + h(y)$ where h is an unknown function of y. Now $v_y = 3x^2 + h'(y) = u_x = 3x^2 - 3y^2$. Thus $h'(y) = -3y^2$. Hence $h(y) = -y^3 + C$. Thus $v(x,y) = 3x^2 - y^3 + C$.