Complex Analysis Tutorial Seven Solutions

Question One.
(a) We have the general Cauchy integral formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

Let $f(z) = e^{3z}$ and $n = 1$. Then we have

$$\int_{C_R} \frac{e^{3z}}{(z-1)^2}dz = 2\pi i f'(1) = 6\pi i e^3.$$

(b)

$$\int_{C_R} \frac{e^{3z}}{(z^2-1)}dz = \int_{C_R} \frac{e^{3z}}{(z+1)(z-1)}dz.$$

Then put

$$f(z) = \frac{e^{3z}}{z+1}.$$

If $R < 2$ then only $z = 1$ is inside C_R. So

$$\int_{C_R} \frac{e^{3z}}{(z^2-1)}dz = \int_{C_R} \frac{e^{3z}}{(z+1)(z-1)}dz$$
$$= 2\pi i f(1) = \pi i e^3.$$

If R is large enough to contain both poles we have by partial fractions

$$\int_{C_R} \frac{e^{3z}}{(z^2-1)}dz = \frac{1}{2}\int_{C_R} e^{3z}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]dz$$
$$= \frac{1}{2} \times 2\pi i(f(1) - f(-1)) = 2\pi i \sinh 3.$$

where $f(z) = e^{3z}$.

Question Two. (a)

$$e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!}.$$

(b)

$$\sin(3z) = \sum_{n=0}^{\infty} (-1)^n \frac{(3z)^{2n+1}}{(2n+1)!}.$$

(c)

$$(z+1)\cos z = (z+1)\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$$
$$= \sum_{n=0}^{\infty} a_n z^n$$

where $a_{2n} = \frac{(-1)^n}{(2n)!}$ and $a_{2n+1} = \frac{(-1)^n}{(2n)!}$.

Question Three.

$$\frac{\cosh z}{z^3} = \frac{1}{z^3}\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right)$$
$$= \frac{1}{z^3} + \frac{1}{2z} + \frac{z}{4!} + \frac{z^3}{6!} + \cdots$$

There is a pole of order 3 at $z = 0$. The residue is $1/2$.

Question Four.

$$\frac{1}{2-z} = \frac{1}{2(1-z/2)} = \frac{1}{2}(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots$$

This is convergent for $|z| < 2$.

Question Five.

$$\frac{z}{1-z} = z(1 + z + z^2 + z^3 + \cdots) = \sum_{n=0}^{\infty} z^{n+1}.$$

Now let $a_n = z^{n+1}$. We use the ratio test first. So

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{z^{n+2}}{z^{n+1}}$$
$$= \lim_{n\to\infty}|z| < 1,$$

provided $|z| < 1$. Alternatively $|a_n|^{1/n} = |z|^{1+\frac{1}{n}}$. So $\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}|z|^{1+\frac{1}{n}} = |z| < 1$ provided $|z| < 1$. So the Taylor series is convergent for $|z| < 1$.

Question Six.

This was done in lectures. Consider the more general problem

$$\frac{e^z}{z^n} = \frac{1}{z^n} + \frac{1}{z^{n-1}} + \cdots$$
$$= \sum_{n=0}^{\infty}\frac{z^{k-n}}{k!}.$$

In general we have a pole of order n at $z = 0$. The residue is $1/(n-1)!$

Question Seven.
(a) We factorise the denominator to obtain

$$\begin{aligned} f(z) &= \frac{z}{(z-2)(z+1)} = \frac{(z+1)-1}{((z+1)-3)(z+1)} \\ &= \frac{1-(z+1)}{(3-(z+1))(z+1)} = \frac{1}{z+1}\frac{1-(z+1)}{(3-(z+1))} \\ &= \frac{1}{z+1}\frac{1}{3}\frac{1-(z+1)}{(1-(z+1)/3)} \\ &= \frac{1}{3(z+1)}(1-(z+1))\left(1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3+\cdots\right) \end{aligned}$$

and this converges for $|z+1| < 3$.
(b)
We can write by partial fractions

$$\begin{aligned} f(z) &= \frac{2}{3}\frac{1}{z-2}+\frac{1}{3}\frac{1}{z+1} \\ &= \frac{-2}{3}\frac{1}{2(1-z/2)}+\frac{1}{3}\frac{1}{z(1+1/z)} \\ &= -\frac{1}{3}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right) \\ &+ \frac{1}{3z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right) \end{aligned}$$

The first series converges for $|z| < 2$ and the second converges for $|z| > 1$, so the resulting series converges on the *overlap*. That is $1 < |z| < 2$.

Question Eight.

$$\begin{aligned} f(z) &= \frac{2z}{1+z^2} = \frac{2z}{(z+i)(z-i)} = \frac{2(z-i)+2i}{(z-i+2i)(z-i)} \\ &= \frac{2}{z-i}\frac{(z-i)+i}{(z-i+2i)} = \frac{2((z-i)+i)}{z-i}\frac{1}{(z-i+2i)} \\ &= \frac{2((z-i)+i)}{2i(z-i)}\frac{1}{1+\frac{z-i}{2i}} \\ &= \frac{2((z-i)+i)}{2i(z-i)}\left(1+\frac{z-i}{2i}+\left(\frac{z-i}{2i}\right)^2+\left(\frac{z-i}{2i}\right)^3+\cdots\right) \\ &= \frac{1}{z-i}-\frac{i}{2}+\frac{1}{2}(z-i)+\cdots \end{aligned}$$

after some tedious algebra. So we have a pole of order 1 and the residue is 1. Note that we usually only need a few terms to identify the order of the pole and find the residue.

Question Nine.

This is just a matter of multiplying terms out until you get bored.

$$f(z) = \frac{\cos(3z)\sin(2z)}{z^3} = \frac{1}{z^3}\left(1 - \frac{(3z)^2}{2!} + \frac{(3z)^4}{4!} + \cdots\right)$$
$$\times \left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots\right)$$
$$= \frac{2}{z^2} - \frac{31}{3} + \frac{781z^2}{60} - \frac{19531z^4}{2520} + \frac{488281z^6}{181440} + \cdots$$

There is a pole or order 2 at $z = 0$. The residue is zero.

$$f(z) = \frac{\cos(3z)\sin(2z)}{z}$$
$$= 2 - \frac{31z^2}{3} + \frac{781z^4}{60} - \frac{19531z^6}{2520} + \cdots$$

Notice that this has NO pole. The point $z = 0$ is a removable singularity. We can define $f(0) = 2$ and the resulting function is continuous.