Differential Equations

Problem sheet one.

(1) Find the general solution of the following ordinary differential equations. For those equations which have an initial condition given, find the unique solution satisfying the given data.

(a) $\dfrac{dy}{dx} = 2x^2y,\ y(0) = 2.$

(b) $\dfrac{dy}{dx} = 2xy^2$

(c) $\dfrac{dy}{dx} = \dfrac{\cos x}{3y^2 + e^y},\ y(0) = 2.$

(d) $2\dfrac{dy}{dx} = y(y-2).$

(e) $2(y-1)\dfrac{dy}{dx} = e^x,\ y(0) = -2.$

(2) Find an integrating factor to solve the following first order linear equations.

(a) $xy' + 2y = 4x^2, y(1) = 4.$

(b) $xy' + (x-2)y = 3x^3e^{-x}$

(c) $y' + (\cot x)y = 3\sin x\cos x.$

(d) $x(\ln x)y' + y = 2\ln x,\ y(e) = 1.$

(e) $y' + 3x^2y = x^2 + e^{-x^3}.$

(3) A Bernoulli equation has the general form

$$y' + p(x)y = q(x)y^n,$$

where n is any number other than 0 or 1. These can be turned into first order linear equations by first dividing both sides of the equation by y^n, then introducing the new variable $u = y^{1-n}$. Use the appropriate substitutions to solve the following Bernoulli equations.

(a) $y' + \dfrac{3}{x}y = x^2y^2.$

(b) $2y' + \dfrac{1}{x+1}y + 2(x^2-1)y^3 = 0.$

(c) $xyy' = y^2 - x^2.$

Problem sheet two.

(1) Obtain the general solution of the following second order ODES.

(a) $2y'' + 4y' + 8y = 0$.

(b) $y'' + 16y = 4x$.

(c) $y'' - 3y' + 2y = 6e^{-x}$.

(d) $y'' + 2y' + 5y = 4e^{-x}\cos 2x$.

(e) $2x^2y'' - 5xy' + 3y = 0$. (Hint: Try $y = x^a$ and show that a must satisfy a quadratic.)

(2) Solve the following third order ODEs.

(a) $y''' - 6y'' + 11y' - 6y = 0$.

(b) $y''' - 4y' = 0$.

(3) Find changes of variables which convert the following Riccati equations to second order linear equations. You do not have to solve the resulting equations.
(a) $f' + \frac{1}{2}f^2 = 2x + 4$

(b) $xf' + 2f^2 + 3xf = 0$. (c) $(1 + x^2)f' + 4f^2 = \sin x$.

(4) Solve the following exact equations.
(a) $2x\sin y - y\sin x + (x^2\cos y + \cos x)y' = 0$.

(b) $(2xy^3 + 8x)dx + (3x^2y^2 + 5)dy = 0, y(2) = -1$,

(c) $(x^2e^y + 3e^x)y' + (2xe^y + 3ye^x) = 0, y(0) = 1/2$.

(d) $y\cos x dx + (\sin x - \sin y)dy = 0$.

(5) Convert the following problems to separable equations and hence solve.
(a) $(y^2 - xy)dx + x^2dy = 0$, (b) $(x + 3y)dx + xdy = 0$

(c) $y' = \dfrac{x^3}{4x^3 - 3x^2y}$, (d) $\dfrac{dy}{dx} = \dfrac{x^3y}{x^4 + y^4}$.
(e) $e^{y/x}y' = 2(e^{y/x} - 1) + \dfrac{y}{x}e^{y/x}$.

Differential Equations

Problem sheet three.

(1) Use the given solution of the homogeneous problem for the following ODEs to construct the general solution.

(a) $x^3y'' + xy' - y = 0,\ y_1 = x.$

(b) $xy'' + (1-2x)y' + (x-1)y = 0,\ y_1 = e^x.$

(c) $2xy'' + (1-4x)y' + (2x-1)y = e^x,\ y_1 = e^x.$

(d) $x^2(x+2)y'' + 2xy' - 2y = (x+2)^2,\ y = x.$

(2) Use variation of parameters to solve the following second order inhomogenous ODEs

(a) $y'' - 3y' + 2y = -\dfrac{e^{2x}}{e^x+1}.$

(b) $y'' + y = \tan x \sec x.$

(c) $y'' + 2y' + y = e^{-x}\sec^2 x.$

(d) $y'' - y = \dfrac{2}{e^x+1}.$

(e) $y'' + 2y' + y = 4e^{-x}\ln x.$

(f) $y'' + y = \mathrm{cosec} x.$

(g) $y'' - 2ay' + (a^2+b^2)y = e^{ax}(A\cos(bx) + B\sin(bx))$

(3) Let a, b be positive real numbers with $a \neq b$. Use Variation of parameters to show that the general solution of the second order ODE

$$y'' + (a+b)y' + aby = F(x),$$

can be written as

$$y(x) = c_1e^{-ax} + c_2e^{-bx} + \frac{1}{b-a}\int_{x_0}^{x}[e^{-a(x-t)} - e^{-b(x-t)}]F(t)dt,$$

where x_0 is some initial point.

(4) If y_1, y_2 are linearly independent solutions of the ODE $Ly = y'' + p(x)y' + q(x)y = 0$, write a general formula for the solution of $Ly = R(x)$, similar to the previous question.

(5) Variation of parameters can be extended to higher order equations. Suppose we want to solve

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x).$$

We assume that we have n linearly independent solutions of the homogenous equation. Call these solutions $y_1, ..., y_n$. The Wronskian is $W(y_1, ..., y_n)$. By analogy with the $n = 2$ case, suppose that there is a particular solution

$$y_p = v_1 y_1 + \cdots + v_n y_n.$$

An extension of the argument used in the $n = 2$ case gives the system

$$\begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \cdots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $y_i^{(k)}$ is the kth derivative of y_i. Cramer's rule gives

$$v_k'(x) = \frac{g(x)W_k(x)}{W(y_1, ..., y_n)},$$

where $W_k(x) = (-1)^{n-k} W(y_1, ... y_{k-1}, y_{k+1}, ..., y_n)$. For example, in the $n = 4$ case,

$$W_1(x) = (-1)^{4-1} \det \begin{pmatrix} y_2 & y_3 & y_4 \\ y_2' & y_3' & y_4' \\ y_2'' & y_3'' & y_4'' \end{pmatrix}, \; W_2(x) = (-1)^{4-2} \det \begin{pmatrix} y_1 & y_3 & y_4 \\ y_1' & y_3' & y_4' \\ y_1'' & y_3'' & y_4'' \end{pmatrix}$$

etc.

(a) Solve the equation $x^3 y''' + x^2 y'' - 2xy' + 2y = x^3 \sin x$. (For the homogeneous equation, try $y = x^a$ as a solution to obtain a cubic for a.)

(b) In structural engineering the bending of a beam under a load $g(x)$ is given by $y^{(4)} - k^2 y'' = g(x),\ 0 < x < L$. Here y is the deflection of the beam from parallel due to the load $g(x)$ and k is a constant and L is the length of the beam. Show that the general solution is given by

$$y(x) = c_1 + c_2 x + c_3 e^{kx} + c_4 e^{-kx} + \int_0^x g(t) G(x,t) dt,$$

where

$$G(x,t) = \frac{t - x}{k^2} - \frac{\sinh(k(t - x))}{k^3}.$$

(6) Show that if u is a known solution of the ODE
$$y'' + p(x)y' + q(x)y = 0,$$
then the substitution $y = uv$ produces a first order DE for v. Solve this to obtain a new proof of the result from class.

Differential Equations

Problem sheet four.

(1) Obtain series solutions for the following differential equations.

(a) $2y'' - xy' - 2y = 0$.

(b) $y'' - (x+1)y = 0$.

(c) $y'' - x^2y' - 2xy = 0$.

(d) $(1+x)y'' - y = 0$.

(e) $y'' - (\sin x)y = 0$.

(f) $y'' - xy' - x^2y = 0$.

(g) $y'' - (x^2+1)y = 0$.

(h) $(x^2+1)y'' - xy' + y = 0$.

(i) $y'' + e^xy' + (1+x^2)y = 0, y(0) = 1, y'(0) = 0$.

Differential Equations

Problem sheet five.

(1) Use the method of Frobenius to solve the following differential equations.

(a) $x^2y'' - 5xy' + (3-x)y = 0$.

(b) $2x^2y'' + xy' - (2x+1)y = 0$.

(c) $2xy'' + 3y' - xy = 0$.

(d) $(2x^2 - x^3)y'' + (7x - 6x^2)y' + (3-6x)y = 0$.

(e) $(2x - 2x^2)y'' + (1+x)y' + 2y = 0$.

(f) $x^2(x+2)y'' - xy' + (1+x)y = 0$.

(h) $3x^2y'' + 8xy' + (x-2)y = 0$.

(i) $x^2y'' - x(1+x)y' + y = 0$.

(2) Express the solutions of the following equations in terms of Bessel functions.

(a) $y'' + x^2y = 0$.

(c) $x^2y'' + 5xy' + (3+4x^2)y = 0$.

(c) $xy'' - 3y' - 9x^5y = 0$.

(d) $x^2y'' + 5xy' + \left(8 + \frac{4}{x^4}\right)y = 0$.

(3) Can any of the equations in question 2 be solved in terms of Bessel functions?

(4) Solve the equation $xy'' + (1-x)y' + ny = 0$. Show that if n is a nonnegative integer, then the equation has solutions which are polynomials of degree n. These are called Laguerre polynomials.

(5) In quantum mechanics we are interested in obtaining the so called wave function for an elementary particle, which we denote ψ. If Ω is a region of space, the probability that the particle will be found in Ω is given by the integral $\int_\Omega |\psi|^2 dx$. The Hydrogen atom consists of a single proton orbited by a single electron.

In a simplified model we assume that the wave function of the electron depends only on the distance r from the proton. In this case

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}(r\psi) - \frac{e_0^2}{r}\psi = E\psi.$$

Here m is the mass of electron, e_0 is the charge on the electron, $E < 0$ is a constant and $\hbar = h/(2\pi)$ where h is known as Planck's constant. Let

$$r = \frac{h^2}{4\pi^2 m e_0^2}x, \; E = \frac{2\pi^2 m e_0^4}{h^2}\epsilon, \; f = x\psi.$$

Now obtain a series solution for the resulting equation.

Differential Equations

Problem sheet six.

(1) Solve Legendre's differential equation
$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$
Show that if n is a positive integer it has solutions which are polynomials. These are known as Legendre polynomials.

(2) Find one solution of Gauss' Hypergeometric differential equation
$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0.$$
(The solutions are special functions known as hypergeometric functions).

(3) Find a solution of the confluent hypergeometric equation
$$xy'' + (b-x)y' - ay = 0.$$

(4) Show that the Hermite equation
$$y'' - 2xy' + 2ny = 0$$
has polynomial solutions when n is a positive integer. These are called Hermite polynomials and denoted $H_n(x)$. Show that these are generated by Rodriguez's formula
$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$
To really impress your friends, prove the Taylor series expansion
$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

(5) Use the definition of the first kind Bessel functions and the series expansions of $\sin x$ and $\cos x$ to prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

(6) Show that $xJ_p'(x) = pJ_p(x) - xJ_{p+1}(x)$. It can also be shown that $xJ_p'(x) = -pJ_p(x) + xJ_{p-1}(x)$. Conclude that
$$\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x).$$

(7) Consider the general second order differential equation
$$u''(x) + a_1(x)u'(x) + a_2(x)u = 0 \quad (1)$$

Show that the change of variable $u = \tilde{u}(x)e^{\int \phi(x)dx}$, where $\phi(x) = -a_1(x)/2$, converts equation (1) into the second order equation

$$\tilde{u}''(x) + [a_2(x) + a_1(x)\phi(x) + \phi'(x) + \phi(x)^2]\tilde{u} = 0.$$

(8) Use the change of variables in the previous question to solve the following ODEs

$$u''(x) + 2\cot(x)u'(x) - u(x) = 0$$
$$u''(x) + 2\tan(x)u'(x) + 2\tan^2(x)u(x) = 0$$

Differential Equations

Problem sheet seven.

(1) Compute the Laplace transforms of the following functions
  (a) $f(x) = e^{-5x}$
  (b) $f(x) = x^2 \cos 2x.,$
  (c) $f(x) = -3\cos(4x) + 5e^x \sin 3x$
  (d) $f(x) = J_0(ax).$
  (e) $f(x) = \frac{ae^{-bx} - be^{-ax}}{b-a}.$
  (f) $f(x) = \sin ax \cosh ax - \cos ax \sinh ax.$
  (g) $f(x) = x^2 \sin(x)\cos(2x).$
  (h) $f(x) = \ln(1+x)$. (This is quite hard).

(2) Prove that the inverse Laplace transform is linear.

(3) Calculate the Laplace transform of the following functions.
  (a) $f(t) = t^2 e^{-3t}\cos(4t),$

  (b) $f(t) = \sin(2t)\cos(4t),$

  (c) $f(t) = e^{-3t}(t+3)^2 \sin(3t)\sin(4t)\cos(3t)\cos(4t),$

  (d) $f(t) = H(t-3)\sin(3t-9)\cos(4t-12).$

(4) Calculate the Laplace transform of $J_\alpha(t)$.

(5) Find the Laplace transforms of
  (a) $\displaystyle\int_0^t x^2 e^x dx$
  (b) $e^{2t}\sqrt{t}$
  (c) $\displaystyle\int_0^t \cos^2 u du$

(6) Find the inverse Laplace transforms of the following functions.
  (a) $e^{-s}\dfrac{1}{(s-2)^2}$
  (b) $\dfrac{s}{(s-2)^{3/2}(s^2+1)}$
  (c) $e^{-2s}\dfrac{1}{s^2+9\pi^2}$
  (d) $\dfrac{s}{s^4(s^2+9)}$

(7) Find a function f which has the following Laplace transforms.

(a) $F(s) = \dfrac{2s+7}{(s+2)(s^2+9)}$

(b) $F(s) = \dfrac{s+3}{(s^2+9)(s^2-16)}$

(c) $F(s) = \dfrac{s^2-9}{(s+3)^2+4}$.

(d) $F(s) = \dfrac{1}{s^a} e^{\frac{k}{s}}$, $a > 0$. (Expand in powers of $1/s$ and invert term by term. Then compare the result to the series for a Bessel function).

(8) Find the inverse Laplace transform of

$$F(s) = \frac{1}{(1+2st)^{\frac{n}{2}}} \exp\left(-\frac{sx}{1+2st}\right).$$

Differential Equations

Problem sheet eight.

(1) Solve the following initial value problems by Laplace transforms.

(a)

$$x''(t) + 4x(t) = 5e^{-t},$$
$$x(0) = 2, x'(0) = 3.$$

(b)

$$x'' + 2x' + x = 4\sin t,$$
$$x(0) = -2, x'(0) = 1$$

(c)

$$x'' + 4x' + 5x = 25t$$
$$x(0) = 0, x'(0) = 2.$$

(2) Solve the initial value problem

$$x'' + x = f(t)$$
$$x(0) = x'(0) = 0,$$
$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ 1, & t > 1. \end{cases}$$

(3) Solve the initial value problem

$$x'' - x = f(t)$$
$$x(0) = 1, x'(0) = 0,$$
$$f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ t-1, & t > 1. \end{cases}$$

(4) By using the Laplace transform, express the solution of the problem

$$x'' + x = f(t),\ x(0) = 0,\ x'(0) = 1$$

as an integral.

(5) Systems of differential equations can often be solved by Laplace transform. Try to solve the following system by Laplace transform.

$$x' + 2x - 2y = 0$$
$$-x + y' + y = 2e^t$$

$x(0) = 0,\ y(0) = 1$. (Hint: Take the Laplace transform of both equations and obtain a pair of simultaneous equations for $X(s)$ and $Y(s)$. Then invert both transforms).

(6) Take the Laplace transform in t to solve the PDE

$$\frac{\partial^2 u}{\partial x \partial t} + \sin t = 0,\ \ t > 0,$$

with $u(x, 0) = x, u(0, t) = 0$.

(7) Solve the wave equation with source term

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - k\sin(\pi x),\ \ 0 < x < 1, t > 0.$$

Suppose that

$$u(x, 0) = 0, \frac{\partial u}{\partial x}(x, 0) = 0, u(0, 1) = u(1, t) = 0.$$

Differential Equations

Problem Sheet Nine

(1) Calculate the Fourier series for the function

$$f(x) = x(x+1), -\pi < x < \pi,$$

with $f(x+2\pi) = f(x)$.

(2) Compute the Fourier series for $f(x) = 2x^2 - 3x + 2,\ -1 < x < 1$, with $f(x+2) = f(x)$.

(3) Calculate the Fourier cosine series of $f(x) = \sin x$ on the interval $[0, \pi]$.

(4) Calculate the Fourier sine series for $f(x) = \cos x$ on the interval $[0, \pi]$. What is the periodicity of the function?

(5) Find the Fourier sine and cosine series for

$$f(x) = 1 - x,\ 0 < x < 1.$$

What is the periodicity of the function?

(6) Obtain the Fourier series expansion of $f(x) = x - [x]$ on the interval $-2 < x < 2$, $f(x+4) = f(x)$ for all x and $[x]$ is the integer part of x.

(7) A more compact form for the Fourier series is to use exponential functions. We expand f as a series of the form

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x},$$

where

$$\widehat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Compute the exponential Fourier series for $f(x) = x$.

(8) Solve the heat equation subject to the given initial and boundary conditions.

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ 0 < x < 1$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0,\ u(x,0) = f(x).$$

Differential Equations

Problem Sheet Ten

(1) Solve the heat equation subject to the given initial and boundary conditions.

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ 0 < x < 3$$
$$u(0,t) = u(3,t) = 0,\ u(x,0) = (2x-1)(x-3).$$

(2) Solve the wave equation subject to the given initial and boundary conditions.

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ 0 < x < \pi$$
$$u(0,t) = u(\pi,t) = 0,\ u(x,0) = (x-3),\ \frac{\partial u}{\partial t}(x,0) = x^2 - 1.$$

(3) Solve the wave equation subject to the given initial and boundary conditions.

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ 0 < x < 1$$
$$u(0,t) = u(1,t) = 0,\ u(x,0) = 2x-4,\ \frac{\partial u}{\partial t}(x,0) = x^2 - x.$$

(4) Find the Fourier sine and cosine series for the given functions.

(a) $f(x) = x\sin x,\ 0 < x < 2\pi,\ f(x+2\pi) = f(x),$ all $x \in \mathbb{R}$.

(b) $f(x) = 1 - \frac{1}{2}x,\ 0 < x < 1,\ f(x+1) = f(x),$ all $x \in \mathbb{R}$.

(5) For the previous questions, what values does Parseval's Theorem give for the sum of the squares of the Fourier coefficients?

(6) Calculate the Fourier cosine series for $f(x) = x^2, 0 \le x \le \pi$ and hence show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

(7) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$.

(8) Find Fourier cosine and sine series for the function

$$f(x) = \begin{cases} 1-2x, & 0 \le x < 1/2 \\ 1 & 1/2 \le x < 1. \end{cases}$$

To what numerical value do your Fourier series converge at $x = 1/2$? What about at $x = -1/2$?

(9) Solve Laplace's equation on the region $0 < x < a, y \geq 0$ with $a > 0$ and the conditions $u(0, y) = u(a, y) = 0$ for all $y \geq 0$ and $u(x, 0) = f(x)$ for $0 \leq x \leq a$. Assume that u is bounded, so that $|u(x, y)| \leq M$, for some positive constant M.

(10) Solve the problem $u_t = u_{xx} + ku_x$, $0 \leq x \leq a$, $t \geq 0$, with $k, a > 0$ and $u(0, t) = u(a, t) = 0$, $u(x, 0) = f(x)$.