35231 DIFFERENTIAL EQUATIONS LECTURE NOTES

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New South Wales 2007
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Date: February 2025.

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1. An Introduction to Differential Equations

1.1. **What is a Differential Equation?** Suppose that F is a continuous function of $n+2$ variables. An nth order ordinary differential equation, or ODE is an expression of the form

$$F(x, y, y', ..., y^{(n)}) = 0.$$

In other words a differential equation is an equation involving the derivatives of a function and the independent variables. A *partial differential equation* or PDE is a differential equation involving more than one independent variable, so the derivatives involved are partial derivatives.

The basic problem which these notes address is the following: Given F, find y. A simple example that we might be interested in is the following. Find a function y such that

$$y' + y'' + y^2 + x^2 = 0.$$

Here $F(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3 + x_4$. In practice we will not use the F notation, but will just write down the actual equation in terms of the unknown function y and the independent variable x. This equation is second order because the highest derivative in the equation is the second derivative. The equation is said to be non-linear because of the y^2 term.

The variety of differential equations that can arise is essentially infinite, and we can only solve certain types of equations exactly. For most equations we need to find a numerical solution. Some equations may not even have a solution.

Nevertheless, the equations that we can solve have, over the last few centuries, yielded a tremendous amount of information about the world in which we live. Fortuitously, the types of equations that we need to solve for the purposes of modelling real world phenonomena, often fall within the classes of exactly solvable equations.

We will begin by considering the basic ordinary differential equations that are typically encountered in a first year course. We will then move on to consider some more advanced techniques.

Example 1.1. Suppose that we have a particle that is moving with velocity v at time t. The instantaneous rate of change of v is called the acceleration. Assume for the moment that the acceleration of the particle is a constant, a.
Then we have

$$\frac{dv}{dt} = a. \tag{1.1}$$

This is a differential equation for the velocity. To solve, we integrate. Thus

$$v(t) = \int a dt = at + c,$$

where c is a constant of integration. In order to determine the constant c, we require more information about the velocity. For example, suppose we know that the initial velocity is v_0, i.e. the velocity at time $t = 0$ is v_0. This is an *initial condition* for the differential equation (1.1).

Applying the initial condition, we have $v(0) = a \cdot 0 + c = v_0$. Hence $c = v_0$ and

$$v(t) = v_0 + at.$$

We can go further and work out a formula for the displacement. Velocity is the rate of change of displacement so we have

$$\frac{ds}{dt} = v = v_0 + at.$$

Integrating again gives

$$s = v_0 t + \frac{1}{2}at^2 + c_2,$$

where c_2 is another constant of integration. Now we apply an initial condition to work out c_2. If we know that at $t = 0$, $s = s_0$, then we get

$$s = v_0 \cdot 0 + \frac{1}{2}a \cdot 0^2 + c_2 = s_0.$$

So

$$s = s_0 + v_0 t + \frac{1}{2}at^2.$$

The expressions for v and s found here are familiar from high school physics.

These differential equations are rather trivial, but they are nevertheless instructive. There is a phenomenon that we wish to understand quantitatively. We have information about the derivatives of the function describing this phenomenon. Why derivatives? Because derivatives measure rates of change and we can measure these. To obtain the quantitative information, we must determine the function from its derivatives, as well as our knowledge of the initial conditions satisfied by the solution. They are in fact simple examples of so called *initial value problems.*

Definition 1.1. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a continuous function of $n + 1$ variables. Consider the ordinary differential equation

$$f(x, y(x), y'(x), ..., y^{(n)}(x)) = 0, \qquad a \leq x \leq b. \tag{1.2}$$

The problem of determining a solution of (1.2) subject to the conditions $y(a) = y_0, ..., y^{(n-1)}(a) = y_{n-1}$ is known as an Initial Value Problem (or IVP) for the ODE (1.2). The conditions $y(a) = y_0, ..., y^{(n-1)}(a) = y_{n-1}$ are known as the initial conditions. An nth order equation requires $n - 1$ initial conditions to specify the solution completely.

A problem arising in nuclear physics is to estimate the amount of mass remaining in a radioactive substance such as uranium after a given amount of time has elapsed. This is quite elementary and is often done in high school.

Example 1.2. Suppose that we have a radioactive substance, say uranium. Let the mass of the uranium be M. Now it has been observed that uranium naturally decays. The mass of uranium determines the rate at which it decays. More precisely, the rate at which M decays, $\frac{dM}{dt}$ is proportional to M itself. We write this as

$$\frac{dM}{dt} \propto M.$$

This means that there exists a constant of proportionality k such that

$$\frac{dM}{dt} = kM.$$

But since M is decaying, $\frac{dM}{dt} < 0$. This tells us that the constant k has to be negative. In other words, $k = -\lambda$, where $\lambda > 0$.
Thus our equation for the mass of uranium present at a given moment in time is

$$\frac{dM}{dt} = -\lambda M$$
$$M(0) = M_0.$$

Notice that we have specified the initial mass of the uranium present. This is also an initial value problem. The obvious problem is how to solve such an equation. We will see how this is done shortly.

Example 1.3. Next, we consider an equation that arises from Newton's second law of motion, which tells us that the total force acting on a body is equal to the mass of the body multiplied by the acceleration. In other words,

$$F = ma.$$

Suppose that a skydiver is in an aeroplane, and he or she jumps out from a height of x feet. In free fall, the skydiver accelerates due to the gravitational pull of the earth. Gravity acts downwards, so we have

$$F_g = -mg.$$

We know that the acceleration $a = \dfrac{dv}{dt}$. This tells us that

$$m\frac{dv}{dt} = -mg.$$

Now when the diver pulls the ripchord, a new force acts on the skydiver. This is the drag force due to the parachute. How do we measure it? Physical arguments, which need not concern us, suggest that a reasonable assumption is that it is proportional to the square of the velocity, i.e.

$$F_d = kv^2.$$

F_d, the drag force, acts against gravity. So the total force on the skydiver is

$$F_g + F_d = -mg + kv^2.$$

But this means that, by Newton's law the velocity of the skydiver must satisfy

$$m\frac{dv}{dt} = -mg + kv^2,$$

after the parachute has been opened. If the skydiver pulls the ripchord after T_1 seconds, they have a velocity of

$$v(T_1) = -gT_1.$$

This means that the velocity after the ripchord has been pulled must satisfy

$$\begin{aligned} m\frac{dv}{dt} &= -mg + kv^2 \\ v(T_1) &= -gT_1. \end{aligned}$$

Again, we ask how do we solve this equation? All these examples of differential equations can be solved by methods you will learn in this course.

Another type of problem which arises with certain kinds of differential equation involving prescribing the behaviour of the solution on the boundary of a region.

Definition 1.2. Consider the ODE $F(x, y', ..., y^{(n)}) = 0$. A boundary value problem for this ODE is to find a solution y on an interval $[a, b]$ satisfying the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$.

Boundary value problems or BVPs arise in many physical situations. For example, suppose we are studying vibrations in a bridge. So we are looking at the vertical motion of the bridge. We can assume (at least we hope that we can assume), that the motion at the ends of bridge is zero. Then if $y(x)$ represents the vertical displacement of the bridge at the point x and a and b are the end points, the boundary conditions would be $y(a) = y(b) = 0$.

We will focus in this course on initial value problems, but we will turn to BVPs towards the end. These are particularly important for the study of partial differential equations.

1.2. **Linear and Non-Linear Equations.** Broadly speaking, differential equations can be classed as either linear or nonlinear. In a linear equation, the differential operator is a linear operator. More precisely, a differential operator is a transformation that sends a function y to some combination of its derivatives. So if we define

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \tag{1.3}$$

then L is a second order linear differential operator and

$$Ly = ay'' + by' + cy. \tag{1.4}$$

We say that L is linear because for any two twice differentiable functions u and v and any constants a and b, we have

$$L(au + bv) = aL(u) + bL(v).$$

A differential equation of the form $Lu = 0$ is said to be homogeneous. Conversely, an equation of the form $Lu = f(x)$ is said to be inhomogeneous.

An equation $Lu = f(x)$ is said to be linear if L is a linear differential operator. Linear equations have the important feature that if $Lu = 0$ and $Lv = 0$, then $L(au + bv) = 0$. So adding two solutions of a *homogeneous* equation together produces another solution.

If the equation is not homogeneous, this does not work. Clearly if $Lu = f(x)$ and $Lv = f(x)$, then $L(u+v) = f(x)+f(x) = 2f(x)$. Thus $u + v$ is not a solution. However, we shall see later that if L is linear, then we can use solutions of the equation $Lu = 0$ to solve $Lu = f(x)$ for any f.

A differential operator L is said to be non-linear if

$$L(au + bv) \neq aL(u) + bL(v).$$

For example, if

$$Ly = y' + y^2, \tag{1.5}$$

then the operator is non-linear. This is because

$$\begin{aligned} L(u + v) &= u' + v' + (u + v)^2 \\ &= u' + v' + u^2 + 2uv + v^2 \\ &\neq L(u) + L(v). \end{aligned}$$

Thus even for homogeneous non-linear equations, adding two solutions together does not produce a new solution. This makes non-linear equations much harder to study. We will consider only a few types of first order non-linear equations in this course. Higher order non-linear equations are much harder, although there are some higher order non-linear equations which can be solved exactly, such as those of so called Painlève type. We do not consider these here.

1.3. **Existence and Uniqueness of Solutions.** The IVP of Definition 1.1 can be shown to have a solution if the function f satisfies a Lipschitz condition. We can also prove that the solution is unique. In order to complete the proof, we needs some facts from elementary analysis.

An important problem in analysis is to consider the convergence of a sequence of functions $\{f_n\}_{n=1}^{\infty}$, or more simply $\{f_n\}$. There are actually different types of convergence, but we will be concerned only with two.

Definition 1.3. A sequence of functions $\{f_n\}$ is said to converge pointwise to f if for every x, the sequence $\{f_n(x)\}$ converges to $f(x)$. That is, given $\epsilon > 0$ we can find $N > 0$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$.

A stronger form of convergence is uniform convergence.

Definition 1.4. A sequence of functions $\{f_n\}$ converges to f uniformly on $X \subseteq \mathbb{R}$ if given any $\epsilon > 0$ we can find an $N > 0$ such that for all $n \geq N$, $\sup_{x \in X} |f_n(x) - f(x)| < \epsilon$.

The significance of uniform convergence is that the choice of N does not depend on x. The same N works for every x, which is not the case with pointwise convergence. Roughly speaking, the sequences $\{f_n(x)\}$ all converge at the same rate for every x. This uniformity makes them nicely behaved and allows us to manipulate them in a way that is not possible if the convergence is only pointwise.

Uniform convergence has very nice properties. For example,

Theorem 1.5. *Suppose that $\{f_n\}$ is a sequence of continuous functions and $f_n \to f$ uniformly. Then f is continuous.*

If the convergence is only pointwise, the result is not true. For example, the sequence $f_n(x) = x^n$ converges to f where $f(x) = 0$, $x \in [0, 1)$ and 1 if $x = 1$. This limit is not continuous, but each f_n is continuous. The problem is that the convergence is not uniform. Another important result allows us to swap limits and integrals.

Theorem 1.6. *Suppose that $\{f_n\}$ is a sequence of integrable functions that converges to f uniformly on $[a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx.$$

Again, this result will not necessarily hold if the convergence is not uniform. For example, if $f_n(x) = nxe^{-nx^2}$ then

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx.$$

The sequence of functions converges to zero, but does not do so uniformly.

For series, we are interested in the sequence defined by the partial sums $S_N(x) = \sum_{n=1}^{n} f_n(x)$. There is a very simple test to decide whether or not convergence of a series is uniform.

Theorem 1.7 (The Weierstrass M test)**.** *Suppose that $\{f_n\}$ is a sequence of functions on $X \subseteq \mathbb{R}$ such that $|f_n(x)| \leq M_n$ for each n and $x \in X$. If $\sum_{n=1}^{\infty} M_n < \infty$. then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on X.*

Proof. Let $S_N(x) = \sum_{n=1}^{\infty} f_n(x)$ and suppose that $|f_n(x)| \leq M_n$. Then for all $N \geq M$

$$\begin{aligned} |S_N(x) - S_M(x)| &= |\sum_{n=M+1}^{N} f_n(x)| \\ &\leq \sum_{n=M+1}^{N} |f_n(x)| \\ &\leq \sum_{n=M+1}^{N} M_n \to 0, \end{aligned}$$

as $N, M \to \infty$. So the series S_N converges independently of x and hence is uniformly convergent. □

Finally we present a useful tool which will be employed in the proof of the major theorem of this section.

Lemma 1.8. *Suppose that the interval I contains a point x_0. Suppose that w is a continous nonnegative function and that there is a constant $M > 0$ such that*

$$w(x) \leq M \left| \int_{x_0}^{x} w(t)dt \right|.$$

Then w is identically zero on I.

Proof. Let

$$W(x) = \int_{x_0}^{x} w(t)dt, \quad x > x_0,$$

and

$$W(x) = -\int_{x_0}^{x} w(t)dt, \quad x < x_0.$$

Clearly $W(x_0) = 0$ and W is nonnegative. Now suppose that $x \geq x_0$. Then $W'(x) = w(x)$ and we have $W'(x) - MW(x) \leq 0$. Multiplying by $e^{-M(x-x_0)}$ gives

$$e^{-M(x-x_0)}W'(x) - MW(x)W'(x) \leq 0$$

which implies that

$$\frac{d}{dx}[W(x)e^{-M(x-x_0)}] \leq 0.$$

This then gives

$$W(x) \leq W(x_0)e^{M(x-x_0)}.$$

But since $W(x_0) = 0$, we have $W(x) = 0$ for $x \geq x_0$. For $x < x_0$ a similar calculation shows that $W(x) = 0$. But this means that $w(x) = 0$ on I.

□

This is all background material for our first major result. This is on the existence and uniqueness of solutions of initial value problems. We prove the result for first order equations. The proof for higher order equations can be obtained from the first order case, as we will demonstrate later.

Theorem 1.9 (Picard)**.** *Let $f(x, y)$ be jointly continuous in x and y and satisfy a Lipschitz condition on the rectangle*

$$R = \{(x, y) : |x - x_0| \leq a, |y - k| \leq b\}.$$

Suppose also that $|f(x, y)| \leq A$. Then the initial value problem

$$y'(x) = f(x, y(x)), \ y(x_0) = k$$

has a unique solution on the interval $[x_0 - \alpha, x_0 + \alpha]$, where

$$\alpha = \min(a, b/A).$$

This theorem gives us conditions which guarantee that an ODE has a unique solution. It should be obvious that there is no point in trying to solve an ODE if the equation doesn't have a solution. The details of the proof are quite involved.

Proof. The idea is to construct a solution inductively. The Fundamental Theorem of Calculus gives the integral equation

$$y(x) = k + \int_{x_0}^{x} f(t, y(t))dt. \tag{1.6}$$

We then introduce the sequence $\{y_n\}$ by defining $y_0 = k$ and

$$y_n(x) = k + \int_{x_0}^{x} f(t, y_{n-1}(t))dt,$$

$|x - x_0| \leq \alpha$.

Each of these functions is well defined and continuous, since the initial function y_0 is continuous and by induction we can prove that $|y_n(x) - k| \leq b$ for $|x - x_0| \leq \alpha$, which shows that $(x, y_{n-1}(x)) \in R$. Specifically, $|y_0(x) - k| = 0 < b$ for $|x - x_0| \leq \alpha$. Suppose then that $(x, y_k(x)) \in R$ for for $|x - x_0| \leq \alpha$. Then $f(x, y_k(x))$ is defined and continuous and so integrable. So we can write

$$y_{k+1}(x) = k + \int_{x_0}^{x} f(t, y_k(t))dt$$

hence

$$\begin{aligned}|y_{k+1}(x) - k| &\leq \left| \int_{x_0}^{x} f(t, y_k(t)) dt \right| \\ &\leq A|x - x_0| \leq A\alpha \leq b.\end{aligned}$$

So $(x, y_{k+1}(x)) \in R$.

To prove that the sequence converges, we prove that the infinite series

$$y_0 + \sum_{n=1}^{\infty} (y_n - y_{n-1}),$$

converges. The nth partial sum of this series is y_n, so if the series converges, the sequence converges. Actually we prove it converges uniformly and hence the limit is continuous. First note

$$|y_1(x) - k| \leq \int_{x_0}^{x} |f(t, k)|\, dt \leq A|x - x_0|. \tag{1.7}$$

Now let the Lipschitz constant be L and notice that

$$\begin{aligned}y_2(x) &= k + \int_{x_0}^{x} f(t, y_1(t)) dt \\ y_1(x) &= k + \int_{x_0}^{x} f(t, y_0(t)) dt,\end{aligned}$$

so that

$$\begin{aligned}|y_2(x) - y_1(x)| &\leq \int_{x_0}^{x} |f(t, y_1(t)) - f(t, y_0(t))| dt \\ &\leq \int_{x_0}^{x} L|y_1(t) - k| dt \\ &\leq AL|x - x_0|.\end{aligned}$$

If we continue this procedure we see that

$$|y_n(x) - y_{n-1}(x)| \leq \frac{AL^{n-1}|x - x_0|^n}{n!}. \tag{1.8}$$

So we have

$$\begin{aligned}y_0 + \sum_{n=1}^{\infty} (y_n - y_{n-1}) &\leq |k| + \sum_{n=1}^{\infty} \frac{AL^{n-1}|x - x_0|^n}{n!} \\ &= |k| + \frac{A}{L}(e^{AL} - 1).\end{aligned}$$

The Weierstrass M test shows that the series is uniformly convergent, hence $\{y_n\}$ converges and the limit is continuous. We now prove that the limit is a solution of the integral equation. Let the limit be y and consider $|y(x) - y_n(x)|$. This is given by

$$\begin{aligned} \left| y(x) - k - \int_{x_0}^{x} f(t, y(t))dt \right| &\leq |y(x) - y_n(x)| + \\ &\quad \int_{x_0}^{x} |f(t, y_{n-1}(t)) - f(t, y(t))| \, dt \\ &\leq |y(x) - y_n(x)| + L \int_{x_0}^{x} |y_{n-1}(t) - y(t)| dt. \end{aligned}$$

By uniform convergence we can find an integer N such that if $n \geq N$ then

$$|y(x) - y_n(x)| \leq \frac{1}{2} \frac{\epsilon}{1 + \alpha L}$$

for all $|x - x_0| \leq \alpha > 0$. So for $n \geq N$ we obtain

$$\begin{aligned} \left| y(x) - k - \int_{x_0}^{x} f(t, y(t))dt \right| &\leq \frac{1}{2} \frac{\epsilon}{1 + \alpha L} + L \left| \int_{x_0}^{x} \frac{1}{2} \frac{\epsilon}{1 + \alpha L} dt \right| \\ &\leq \frac{1}{2} \frac{\epsilon}{1 + \alpha L} + \frac{1}{2} \frac{\epsilon}{1 + \alpha L} |x - x_0| \\ &\leq \epsilon. \end{aligned}$$

Since this holds for all $\epsilon > 0$ we conclude that

$$y(x) = k + \int_{x_0}^{x} f(t, y(t))dt,$$

so y is a solution of the integral equation and hence the differential equation.

Finally we prove uniqueness. Suppose that

$$\begin{aligned} u(x) &= k + \int_{x_0}^{x} f(t, u(t))dt \\ v(x) &= k + \int_{x_0}^{x} f(t, v(t))dt. \end{aligned}$$

Then

$$|u(x) - v(x)| \leq L \int_{x_0}^{x} |u(t) - v(t)| dt.$$

By Lemma 1.8, this implies that $u = v$. □

Now we turn to the major theme of this course: actually solving differential equations.

Example 1.4. Here is a simple example of an IVP that has no solution. We want to solve

$$xy'' + y' + 3y = 1,$$

subject to the conditions $y(0) = 1, y'(0) = 0$. Clearly this is impossible, since at $x = 0$ we have $0 \times y''(0) + y'(0) + 3y(0) = 3y(0) = 1$, which implies that $y(0) = 1/3$. This contradicts the initial condition, so there is no solution. Problems like this are said to be *ill posed*.

1.4. **First Order Separable Equations.** We begin with a class of equations known as first order separable. Suppose that we have an initial value problem of the form

$$\begin{aligned} \frac{dy}{dt} &= f(y)g(t) \\ y(0) &= y_0. \end{aligned} \tag{1.9}$$

Since the right hand side of (1.9) is a product of a function of y and a function of t, we can separate it. The idea is to rewrite the equation as

$$\frac{dy}{f(y)} = g(t)dt,$$

and then integrate both sides.

Let us consider the uranium problem discussed earlier. We had

$$\begin{aligned} \frac{dM}{dt} &= -\lambda M \\ M(0) &= M_0. \end{aligned}$$

We have a separable equation. So we have

$$\frac{dM}{M} = -\lambda dt.$$

Thus

$$\int \frac{dM}{M} = \int -\lambda dt.$$

Hence $\ln|M| = -\lambda t + c$, where c is a constant of integration. Thus

$$M = e^c e^{-\lambda t},$$

$$M(0) = e^c = M_0.$$

The solution is therefore

$$M(t) = M_0 e^{-\lambda t}.$$

This shows that the quantity of uranium decays exponentially over time. This is a key fact in nuclear physics.

Now let us consider the plight of the skydiver discussed previously.

Example 1.5. We wish to solve the differential equation

$$\begin{aligned} m\frac{dv}{dt} &= -mg + kv^2 \\ v(T_1) &= -gT_1. \end{aligned}$$

We rewrite the equation as

$$\frac{dv}{dt} = \frac{k}{m}v^2 - g.$$

The right hand side is a function of v only. So it is separable. Thus we have

$$\frac{dv}{\frac{k}{m}v^2 - g} = dt. \tag{1.10}$$

The problem here is to integrate the left hand side. We rewrite (1.10) as

$$\frac{m}{k}\int \frac{dv}{(v^2 - \frac{mg}{k})} = \int dt.$$

Now $v^2 - \frac{mg}{k} = (v - \sqrt{\frac{mg}{k}})(v + \sqrt{\frac{mg}{k}})$. (Note $\frac{mg}{k} > 0$.) So, by partial fractions we have

$$\begin{aligned}\frac{m/k}{(v^2 - \frac{mg}{k})} &= \frac{m/k}{(v - \sqrt{\frac{mg}{k}})(v + \sqrt{\frac{mg}{k}})} \\ &= \frac{A}{v - \sqrt{\frac{mg}{k}}} + \frac{B}{v + \sqrt{\frac{mg}{k}}}.\end{aligned}$$

Take $A = \frac{m/k}{2\sqrt{\frac{mg}{k}}}$, $B = -\frac{m/k}{2\sqrt{\frac{mg}{k}}}$. Then

$$\frac{m/k}{2\sqrt{\frac{mg}{k}}}\left(\frac{1}{v - \sqrt{\frac{mg}{k}}} - \frac{1}{v + \sqrt{\frac{mg}{k}}}\right) = \frac{m/k}{2\sqrt{\frac{mg}{k}}}\frac{2\sqrt{\frac{mg}{k}}}{v^2 - mg/k}.$$

Hence

$$\begin{aligned}\frac{m}{k}\int \frac{dv}{v^2 - \frac{mg}{k}} &= \frac{m/k}{2\sqrt{\frac{mg}{k}}}\int \left(\frac{1}{v - \sqrt{\frac{mg}{k}}} - \frac{1}{v + \sqrt{\frac{mg}{k}}}\right) dv \\ &= \frac{m/k}{2\sqrt{\frac{mg}{k}}}\left(\ln\left|v - \sqrt{\frac{mg}{k}}\right| - \ln\left|v + \sqrt{\frac{mg}{k}}\right|\right) + c \\ &= \frac{1}{2}\sqrt{\frac{m}{gk}}\ln\left|\frac{v - \sqrt{\frac{mg}{k}}}{v + \sqrt{\frac{mg}{k}}}\right| + c_1 \\ &= t + c_2,\end{aligned}$$

since $\int dt = t + c_2$ for some constant c_2. Letting $c_2 - c_1 = c$ gives

$$\frac{1}{2}\sqrt{\frac{m}{gk}}\ln\left|\frac{v - \sqrt{\frac{mg}{k}}}{v + \sqrt{\frac{mg}{k}}}\right| = c + t.$$

We now have to solve this for v as a function of t.

$$\ln\left|\frac{v - \sqrt{\frac{mg}{k}}}{v + \sqrt{\frac{mg}{k}}}\right| = 2\sqrt{\frac{gk}{m}}t + c', \quad c' = 2\sqrt{\frac{gk}{m}}c.$$

Taking exponentials, we have

$$\frac{v - \sqrt{\frac{mg}{k}}}{v + \sqrt{\frac{mg}{k}}} = Ae^{\alpha t}, \quad A = e^{c}, \quad \alpha = 2\sqrt{\frac{gk}{m}}.$$

Now $v - \sqrt{\frac{mg}{k}} = \left(v + \sqrt{\frac{mg}{k}}\right) Ae^{\alpha t}$.
Hence

$$v - vAe^{\alpha t} = \sqrt{\frac{mg}{k}} Ae^{\alpha t} + \sqrt{\frac{mg}{k}}$$

and

$$v(t) = \sqrt{\frac{mg}{k}} \left(\frac{(1 + Ae^{\alpha t}}{1 - Ae^{\alpha t}}\right).$$

We now have to fit the initial condition, i.e. find A.

$$v(T_1) = \sqrt{\frac{mg}{k}} \left(\frac{1 + Ae^{\alpha T_1}}{1 - Ae^{\alpha T_1}}\right) = -gT_1.$$

Cross multiplying, we get

$$\begin{aligned} \frac{1 + Ae^{\alpha T_1}}{1 - Ae^{\alpha T_1}} &= -gT_1 \sqrt{\frac{k}{mg}} \\ &= -T_1 \sqrt{\frac{gk}{mg}} \\ &= -2\alpha T_1. \end{aligned}$$

$$\begin{aligned} 1 + Ae^{\alpha T_1} &= \left(1 - Ae^{\alpha T_1}\right)(-2\alpha T_1) \\ &= -2\alpha T_1 + 2\alpha T_1 Ae^{\alpha T_1} \\ 1 + 2\alpha T_1 &= (2\alpha T_1 - 1) Ae^{\alpha T_1}. \end{aligned}$$

So

$$A = \left(\frac{2\alpha T_1 + 1}{2\alpha T_1 - 1}\right) e^{-\alpha T_1}.$$

We have thus solved the initial value problem for the DE.

1.4.1. *A Problem from Biology.* Another important example of a separable differential equation is the logistic equation. This equation arises in many areas, but particularly in population growth.

Example 1.6. Suppose that a population of bacteria is growing in a Petri dish. At $t = 0$ the population is P_0. The rate of production of a population of new bacteria is proportional to the number of bacteria present. Of course bacteria are also dying. So this would suggest a model $\frac{dP}{dt} = kP, \quad P(0) = P_0$, which has a solution $P(t) = P_0 e^{kt}$. But this is not a very realistic model. It suggests that the population grows without bound.

A more realistic model takes into account the fact that the population growth should slow down as some "saturation" level is reached.

Call this level M. So, $\frac{dP}{dt} = 0$ when $P = M$. But for small P, $\frac{dP}{dt} \propto P$. A model which incorporates this feature is

$$\frac{dP}{dt} = k(M-P)P$$
$$P(0) = P_0.$$

This is known as the logistic equation. It is a model that arises in many areas of biology.

$$\frac{dP}{dt} = k(M-P)P, \qquad P(0) = P_0$$
$$\frac{dP}{P(M-P)} = kdt.$$

As before, we have

$$\int \frac{dP}{P(M-P)} = \int kdt$$
$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}, \quad A = \frac{1}{M}, \quad B = \frac{1}{M}.$$

Therefore

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right).$$

Hence

$$\int kdt = kt + c_1$$
$$= \frac{1}{M}\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP$$
$$= \frac{1}{M}\left(\ln|P| - \ln|M-P|\right) + c_1.$$

So that

$$\ln\left|\frac{P}{M-P}\right| = Mkt + c, \qquad c = c_2 - c_1.$$

Taking exponentials of both sides gives

$$\frac{P}{M-P} = Ae^{Mkt},$$

which implies

$$P = A(M-P)e^{Mkt}.$$

We rearrange this to get $APe^{Mkt} + P = AMe^{Mkt}$, or

$$P(t) = \frac{AMe^{Mkt}}{1 + Ae^{Mkt}}.$$

Now $P(0) = P_0$. Hence

$$P(0) = \frac{AM}{1+A} = P_0.$$

So that $AM = P_0(1 + A)$, and therefore

$$A(M - P_0) = P_0,$$

or

$$A = \frac{P_0}{M - P_0}.$$

Thus

$$\begin{aligned} P(t) &= \frac{\frac{MP_0 e^{Mkt}}{M-P_0}}{1 + \frac{P_0 e^{Mkt}}{M-P_0}} \\ &= \frac{MP_0 e^{Mkt}}{M - P_0 + P_0 e^{Mkt}} \\ &= \frac{MP_0 e^{Mkt}}{M + P_0(e^{Mkt} - 1)}. \end{aligned}$$

We can rewrite this as

$$P(t) = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0}.$$

This makes it easier to see the limiting behaviour of the population.

$$\begin{aligned} \lim_{t\to\infty} P(t) &= \lim_{t\to\infty} \left(\frac{MP_0}{(M - P_0)e^{-Mkt} + P_0} \right) \\ &= \frac{MP_0}{P_0} \\ &= M. \end{aligned}$$

In this model, the bacterial population tends to a stable value of M.

Example 1.7. Consider the equation

$$\frac{dy}{dt} = \frac{t^2}{\cos y + \sin y}.$$

This is separable, so we write

$$(\cos y + \sin y)dy = t^2 dt$$

and

$$\int (\cos y + \sin y)dy = \int t^2 dt$$

So the solution $y(t)$ is given implicitly by

$$\cos(y(t)) - \sin(y(t)) = -\frac{1}{3}t^3 + c. \tag{1.11}$$

(Combining integration constants of both sides.) Solving this for y in terms of t is actually quite straightforward. Observe that

$$\sqrt{2}\cos\left(y(t)+\frac{\pi}{4}\right)=\cos(y(t))-\sin(y(t)).$$

Thus the solution is

$$y(t)=\cos^{-1}\left(-\frac{t^3}{3\sqrt{2}}+C\right)-\frac{\pi}{4}. \tag{1.12}$$

1.4.2. *Solutions Given Implicitly.* First order separable equations are very common. Whether we can solve them depends very much on whether we can do the necessary integration. Having performed the integration, we would like to express the solution y as an explicit solution of the independent variable, but we cannot always do this.

Example 1.8. Solve the equation

$$\frac{dy}{dx}=(x^2+3)(y^2+y^3).$$

Separating variables leads to

$$\frac{dy}{y^2+y^3}=(x^2+3)dx.$$

This is the same as

$$\int\left(\frac{1}{y^2}+\frac{1}{y+1}-\frac{1}{y}\right)dy=\int(x^2+3)dx.$$

Integration then gives

$$-\frac{1}{y}+\ln\left(\frac{1+y}{y}\right)=\frac{1}{3}x^3+3x+C.$$

We can rewrite this as

$$\left(\frac{1+y}{y}\right)e^{-1/y}=Ae^{\frac{1}{3}x^3+3x}, \tag{1.13}$$

$A=e^C$. This is as far as we can go. Writing the solution in the form $y(x)=f(x)$ does not seem possible for this example. To determine y for a specific value of x, we would have to solve (1.13) numerically, for example with Newton's method.

What we have in the previous example is a solution given *implicitly.* That is, we have an expression connecting the solution y and the independent variable that does not involve any derivatives. In other words we have an expression of the form $F(y(x)),x)=C$ for some function F. This is typical for nonlinear differential equations. Usually we cannot solve the equation $F(y,x)=C$ explicitly for y in terms of x, but instead have to solve it numerically. Thus $F(y,x)=C$ is known as an *implicit solution.*

1.5. **First order linear equations.** These equations have the general form

$$a(x)\frac{dy}{dx} + b(x)y = f(x)$$

If $a(x) \neq 0$, then we can rewrite the equation as

$$\frac{dy}{dx} + \frac{b(x)}{a(x)}y = \frac{f(x)}{a(x)}$$

which is the same as

$$\frac{dy}{dx} + c(x)y = g(x), \tag{1.14}$$

where $\frac{b(x)}{a(x)} = c(x)$ and $\frac{f(x)}{a(x)} = g(x)$.

To solve a first order linear equation, the first step is to write it in the form (1.14). Now we make the following important observation. Recall that the fundamental Theorem of Calculus states that for any integrable function $c(t)$,

$$\frac{d}{dx}\int_a^x c(t)dt = c(x).$$

Thus

$$\frac{d}{dx}\exp\left(\int_a^x c(t)dt\right) = c(x)e^{\int_a^x c(t)dt}.$$

The reason why this is important is that the right hand side of (1.14) is "almost" a derivative.

The product rule of differentiation says that

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

So what does this tell us in the context of our ODE? The right hand side is

$$\frac{dy}{dx} + c(x)y.$$

We are going to turn this into a *single* derivative in the following way. Multiply y by $e^{\int c(x)dx}$, then take the derivative to get

$$\frac{d}{dx}\left(y(x)e^{\int c(x)dx}\right) = e^{\int c(x)dx}\frac{dy}{dx} + c(x)e^{\int c(x)dx}y, \tag{1.15}$$

from the product rule and the fact that $\frac{d}{dx}e^{\int c(x)dx} = c(x)e^{\int c(x)dx}$. So if we multiply (1.14) by $e^{\int c(x)dx}$ we get

$$e^{\int c(x)dx}\frac{dy}{dx} + c(x)e^{\int c(x)dx}y = g(x)e^{\int c(x)dx}.$$

But by (1.15), this is

$$\frac{d}{dx}\left(e^{\int c(x)dx}y\right) = g(x)e^{\int c(x)dx}.$$

Integrating both sides of the equation gives

$$e^{\int c(x)dx}y = \int g(x)e^{\int c(x)dx}dx,$$

or

$$y = e^{-\int c(x)dx}\int g(x)e^{\int c(x)dx}dx.$$

In practice, we rarely use this formula. It is easier to just follow the procedure from the beginning. The function $e^{\int c(x)dx}$ is called an *integrating factor.*

Example 1.9. Let a, b be constants, $a \neq 0$. Solve

$$\frac{dy}{dx} + ay = b$$
$$y(0) = y_0.$$

The integrating factor is $e^{\int a dt} = e^{at}$. Multiplying the equation by e^{at} gives

$$e^{at}\frac{dy}{dx} + ae^{at}y = be^{at}.$$

Therefore

$$\frac{d}{dt}\left(e^{at}y\right) = be^{at}.$$

We integrate both sides

$$\int \frac{d}{dt}\left(e^{at}y\right)dt = \int be^{at}dt.$$

So

$$e^{at}y = \frac{b}{a}e^{at} + C.$$

Then

$$y = Ce^{-at} + \frac{b}{a}$$

and the initial condition gives

$$y(0) = C + \frac{b}{a} = y_0 \quad \Rightarrow C = y_0 - \frac{b}{a}.$$

Hence the solution is

$$y = \left(y_0 - \frac{b}{a}\right)e^{-at} + \frac{b}{a}.$$

Example 1.10. As our next example, consider

$$x\frac{dy}{dx} - ky = x^2,$$

k a constant. First we divide by x to put the equation in the appropriate form.

$$\frac{dy}{dx} - \frac{k}{x}y = x.$$

The integrating factor is

$$e^{-\int \frac{k}{x}} = e^{-k\ln x} = e^{\ln(x^{-k})} = x^{-k}.$$

So we have

$$x^{-k}\frac{dy}{dx} - \frac{k}{x}x^{-k}y = x^{-k+1}.$$

Hence

$$\frac{d}{dx}\left(x^{-k}y\right) = x^{-k+1}.$$

We have 2 cases. First, $k \neq 2$. Then $1-k \neq -1$. Integration gives

$$x^{-k}y = x^{-k+2} + C.$$

So we have established that

$$y = x^2 + Cx^k.$$

If $k = 2$, then $1 - k = -1$. So integration gives

$$x^{-2}y = \int \frac{dx}{x} = \ln x + C.$$

The solution is thus

$$y = x^2\ln x + Cx^2.$$

Example 1.11. For a third example, consider

$$\frac{dy}{dx} - \tan xy = x.$$

The integrating factor is

$$e^{-\int \tan x dx}.$$

$$\begin{aligned} -\int \tan x dx &= -\int \frac{\sin x}{\cos x}dx \\ &= \int \frac{du}{u}, \quad u = \cos x \\ &= \ln\left|\cos x\right|. \end{aligned}$$

Thus

$$e^{-\int \tan x dx} = e^{\ln|\cos x|} = \cos x.$$

Hence

$$\cos x\frac{dy}{dx} - \cos x \tan xy = x\cos x,$$

or

$$\frac{d}{dx}\left((\cos x)\, y\right) = x\cos x,$$

and integration gives

$$\begin{aligned}(\cos x)\, y &= \int x \cos x dx \\ &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C.\end{aligned}$$

So

$$\begin{aligned}y &= \frac{x \sin x + \cos x + C}{\cos x} \\ &= x \tan x + 1 + C \sec x.\end{aligned}$$

Perhaps the most powerful tool for solving ODEs is the change of variable, in much the same way that we can use changes of variables to integrate difficult functions. Here we give some examples of how this works. Many other cases are known.

1.5.1. *Bernoulli Equations.* A Bernoulli equation is one of the form

$$y' + p(x)y = q(x)y^n. \tag{1.16}$$

Dividing through by y^n produces the equation

$$y'y^{-n} + p(x)y^{1-n} = q(x).$$

This suggests the change of variables $u = y^{1-n}$. Differentiation gives

$$u' = (1-n)y'y^{-n}. \tag{1.17}$$

Consequently (1.16) becomes

$$\frac{1}{1-n}u' + p(x)u = q(x). \tag{1.18}$$

This is a first order linear equation.

Example 1.12. We solve $y' + \frac{1}{3x}y = x^2y^4$. Here $n = 4$ so the change of variables $u = y^{-3}$ leads to

$$u' - \frac{1}{x}u = -3x^2.$$

The integrating factor is $1/x$ so

$$\left(\frac{1}{x}u\right)' = -3x. \tag{1.19}$$

Hence $\frac{1}{x}u = -\frac{3}{2}x^2 + C$. Or $u = -\frac{3}{2}x^3 + Cx$. Finally $y = u^{-1/3}$ or

$$y = (-\frac{3}{2}x^3 + Cx)^{-1/3}.$$

1.6. **Riccati Equations.** Nonlinear equations are often very difficult to solve, but there are some which can be handled very effectively. Riccati equations are named after Count Jacopo Francesco Riccati (1676-1754), who studied them extensively. They turn up in many applications. Importantly, they can all be linearised.

A Riccati equation has the form

$$m(x)y' + a(x)y + b(x)y^2 = c(x). \tag{1.20}$$

If $c(x) = 0$ then we can also regard this as a Bernoulli equation. As an example

$$xy' - y + \frac{1}{2}y^2 = Ax^2 + Bx + C$$

is a Riccati equation. Any Riccati equation can be turned into a second order linear ODE. The trick is to make a change of variables. By dividing through by m we can make the coefficient of y' equal to 1, so we may as well set $m = 1$ and we do not lose any generality. Now define

$$y = A(x)f'/f,$$

where A is to be determined. Differentiating gives

$$y' = A'(x)\frac{f'}{f} + A(x)\left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2\right). \tag{1.21}$$

This is substituted into (1.20) to give

$$A'(x)\frac{f'}{f} + A(x)\left(\frac{f''}{f} - \left(\frac{f'}{f}\right)^2\right) + a(x)A(x)\frac{f'}{f} + b(x)A^2(x)\left(\frac{f'}{f}\right)^2 = c(x)$$

Observe that if $-A(x)+b(x)A^2(x) = 0$ then the nonlinear terms disappears. This happens if $A(x)b(x) = 1$. We are then left with the second order linear equation

$$A(x)f'' + (A'(x) + a(x)A(x))f' - c(x)f(x) = 0.$$

Example 1.13. The equation $xy' - y + \frac{1}{2}y^2 = Ax^2 + Bx + C$ is linearised by putting $y' = 2xf'/f$. This produces the equation

$$2x^2f'' - (Ax^2 + Bx + C)f = 0.$$

Methods for solving equations of this type will be discussed later.

Riccati equations have some very interesting properties. We will not give a detailed discussion, but one is well worth mentioning. The difference between nonlinear and linear equations is that adding solutions of linear equations together produces new solutions. Adding solutions of nonlinear equations together does not produce a new solution. However for some nonlinear equations, there are ways of combining solutions together to get new solutions. These are called nonlinear superposition principles.

Riccati equations possess a nonlinear superposition principle that was discovered by the Norwegian mathematician Sophus Lie (which is pronounced Lee) and independently by Eduard Weyr in 1875. Suppose that y_1, y_2, y_3 satisfy a Riccati equation. Let a be a constant. Then

$$y_4 = \frac{y_1(y_3 - y_2) + ay_2(y_1 - y_3)}{y_3 - y_2 + a(y_1 - y_3)}, \tag{1.22}$$

is also a solution of the same Riccati equation. In this way we can produce chains of solutions of Riccati equations and introduce arbitrary parameters into the solutions.

Example 1.14. Three solutions of the Riccati equation $xf' - f + \frac{1}{2}f^2 = Ax + B$ are

$$f(x) = \frac{1}{2} + b\sqrt{x},\ g(x) = \frac{1}{2} + b\sqrt{x}\tanh(b\sqrt{x}),\ h(x) = \frac{1}{2} + b\sqrt{x}\coth(b\sqrt{x}),$$

where we set $B = -\frac{3}{8}, A = \frac{1}{2}b^2$. Using these solutions and the nonlinear superposition principle we easily generate the fourth solution $y(x) =$

$$\frac{2b\sqrt{x}(a - \coth(b\sqrt{x}) - a + \mu\coth(b\sqrt{x}) + (1 - 2\mu b\sqrt{x})\tanh(b\sqrt{x})}{2\left(-a + \mu\coth(b\sqrt{x}) + \tanh(b\sqrt{x})\right)},$$

with $\mu = a - 1$. Using y and two of f, g, h we can generate solutions y_2, y_3, y_4 etc.

1.7. **Exact Equations.**

There are many types of first order equations which can be solved explicitly. Exact equations are of special interest because of a remarkable fact discovered by Sophus Lie, which we briefly mention below.

We consider an ODE written in the form

$$P(x, y)dx + Q(x, y)dy = 0. \tag{1.23}$$

This is equivalent to the equation $Q(x, y)y' + P(x, y) = 0$. These are easy to solve, provided that P and Q satisfy a special condition.

Theorem 1.10. *If*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y},$$

then there is a function $F(x, y)$ such that

$$Q = \frac{\partial F}{\partial y},\ P = \frac{\partial F}{\partial x},$$

and the solution of the exact equation

$$P(x, y)dx + Q(x, y)dy = 0, \tag{1.24}$$

is given implicitly by $F(x, y) = C$, with C a constant.

Proof. First suppose that we can find an F such that $F_x = P, F_y = Q$. Now suppose that $F(x, y) = C$. Then

$$\begin{aligned} \frac{d}{dx}F(x, y(x)) &= F_y y' + F_x \\ &= Q(x, y)y' + P(x, y) = 0. \end{aligned}$$

Alternatively, if $Q(x, y)y'+P(x, y) = 0$, then we must have $F(x, y(x)) = C$. Finally observe that

$$F(x, y) = \int_{x_1}^{x} P(t, y)dt + \int_{y_1}^{y} Q(x, t)dt \tag{1.25}$$

satisfies $F_x(x, y) = P(x, y), F_y(x, y) = Q(x, y)$. □

Example 1.15. Solve the differential equation

$$(3x^2y - 2y^3 + 3)dx + (x^3 - 6xy^2 + 2y)dy = 0.$$

The equation is exact because

$$Q_x = 3x^2 - 6y^2 = P_y.$$

So there is an F such that

$$F_x = 3x^2y - 2y^3 + 3, F_y = x^3 - 6xy^2 + 2y.$$

Now

$$F(x, y) = \int F_x dx = x^3y - 2xy^3 + 3x + g(y)$$

where g is an arbitrary function of y. Next differentiate this with respect to y, to obtain

$$F_y = x^3 - 6xy^2 + g'(y).$$

This gives us two expressions for F_y, which must be equal. Comparing we see that $g'(y) = 2y$ so $g(y) = y^2 + D$, and the solution y is given implicitly by

$$x^3y - 2xy^3 + 3x + y^2 = C - D = C',$$

where C is a constant. Note that in practice we can ignore the constant of integration for g, since it will just be combined with C to produce another constant.

The importance of exact equations lies in the fact that essentially all first order equations which we can solve, can be recast as exact equations. Indeed Lie showed that many equations which are not exact can be converted to exact equations by means of an integrating factor.

There is a caveat obviously. Obtaining the integrating factor may be harder than solving the original DE. However it is often possible to find it. This integrating factor comes from the *symmetry group* of the equation. Unfortunately this is well beyond the scope of this course, so we will not discuss it further.

1.8. **Making First Order Equations Separable.** The word homogenous is used in to mean different things in mathematics, which can be somewhat confusing. We will refer to certain kinds of differential equations as homogeneous. A function can also be considered to be homogeneous.

Definition 1.11. A function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree k if $f(tx_1, ..., tx_n) = t^k f(x_1, ..., x_n)$ for every $t > 0$ and all $x_1, ..., x_n$.

As an example, the function $f(x, y, z) = x^2 y^3 z^4$ is homogeneous of degree 9, since

$$f(tx, ty, tz) = t^2 x^2 t^3 y^3 t^4 z^4 = t^9 x^2 y^3 z^4.$$

Now consider the differential equation

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)},$$

where P and Q are homogeneous of the same degree. A simple substitution will make this equation separable.

Proposition 1.12. *If P and Q are homogenous functions of the same degree then the substitution $y = xv(x)$ will make the differential equation*

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}$$

separable.

Proof. Suppose that P and Q are homogeneous of degree n, then

$$P(x, y) = P(x, xv) = x^n P(1, v), Q(x, y) = Q(x, xv) = x^n Q(1, v)$$

and $y' = xv' + v$. So the ODE becomes

$$\begin{aligned} xv' + v &= \frac{x^n P(1, v)}{x^n Q(1, v)} \\ &= \frac{P(1, v)}{Q(1, v)}. \end{aligned}$$

Which is the same as

$$x\frac{dv}{dx} = K(v)$$

where

$$K(v) = \frac{P(1, v)}{Q(1, v)} - v.$$

This equation is separable. □

Example 1.16. Solve the equation

$$\frac{dy}{dx} = \frac{2xy}{x^2 + y^2}.$$

The numerator and denominator are homogeneous of degree 2. Set $y = xv$. Then

$$x\frac{dv}{dx} = \frac{2v}{1+v^2} - v = \frac{v-v^3}{1+v^2}.$$

Thus

$$\frac{1+v^2}{v-v^3}dv = \frac{dx}{x}$$

Using partial fractions we obtain

$$\left(\frac{1}{v} - \frac{1}{v+1} - \frac{1}{v-1}\right) dv = \frac{dx}{x}.$$

Hence

$$\ln v - \ln(v^2-1) = \ln x + C.$$

Taking exponentials gives

$$\frac{v}{v^2-1} = Ax, \;\; A = e^C. \tag{1.26}$$

But $v = y/x$. So that the solution is given implicitly by

$$\frac{y}{y^2-x^2} = A.$$

1.9. **Second Order Linear, Constant Coefficient Equations.** Next we turn to second order linear equations. Our first aim is to solve the equation

$$ay'' + by' + cy = 0, \qquad a \neq 0.$$

Since $a \neq 0$, we may as well divide by a to get

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0.$$

So really we need only consider equations of the form

$$y'' + my' + ny = 0, \qquad m, n \in \mathbb{R},$$

where at least one of m, n is nonzero.

Let us first consider the quadratic equation $\lambda^2 + m\lambda + n = 0$. Suppose this has two roots α and β. In other words,

$$(\lambda - \alpha)(\lambda - \beta) = \lambda^2 - (\alpha + \beta)\lambda + \alpha\beta = \lambda^2 + m\lambda + n.$$

Now we are going to use this to rewrite our differential equation. Notice that

$$\begin{aligned}\left(\frac{d}{dx} - \alpha\right)\left(\frac{dy}{dx} - \beta y\right) &= \frac{d^2y}{dx^2} - \beta\frac{dy}{dx} - \alpha\frac{dy}{dx} + \alpha\beta y \\ &= \frac{d^2y}{dx^2} - (\alpha+\beta)\frac{dy}{dx} + \alpha\beta y \\ &= \frac{d^2y}{dx^2} + m\frac{dy}{dx} + ny. \end{aligned} \tag{1.27}$$

The first step is to now introduce a new function

$$z = \frac{dy}{dx} - \beta y.$$

By (1.27),

$$\begin{aligned}
\frac{d^2y}{dx^2} + m\frac{dy}{dx} + ny &= \left(\frac{d}{dx} - \alpha\right)\left(\frac{dy}{dx} - \beta y\right) \\
&= \left(\frac{d}{dx} - \alpha\right) z \\
&= \frac{dz}{dx} - \alpha z \\
&= 0.
\end{aligned}$$

But the solution of $\frac{dz}{dx} = \alpha z$ is simply $z = Ae^{\alpha x}$.
Consequently, we must have

$$z = \frac{dy}{dx} - \beta y = Ae^{\alpha x}.$$

But this is a first order linear equation. The integrating factor is $e^{-\beta x}$. We have two cases.

Case 1. $\alpha \neq \beta$.
Then

$$e^{-\beta x}\frac{dy}{dx} - \beta e^{-\beta x} y = Ae^{(\alpha-\beta)x}.$$

Which means

$$\frac{d}{dx}\left(e^{-\beta x} y\right) = Ae^{(\alpha-\beta)x}.$$

Thus

$$\begin{aligned}
e^{-\beta x} y &= \frac{A}{\alpha - \beta} e^{(\alpha-\beta)x} + B \\
y &= \frac{A}{\alpha - \beta} e^{\alpha x} + Be^{\beta x} \\
&= A' e^{\alpha x} + Be^{\beta x}, \quad A' = \frac{A}{\alpha - \beta}.
\end{aligned}$$

So if $\alpha \neq \beta$, i.e. $\lambda^2 + m\lambda + n = 0$ has distinct roots α, β, the general solution is

$$y = Ae^{\alpha x} + Be^{\beta x}.$$

Case 2. $\alpha = \beta$. Then $\alpha - \beta = 0$. So

$$\frac{d}{dx}\left(e^{-\beta x}y\right) = A$$
$$\text{i.e.} \quad e^{-\beta x}y = Ax + B$$
$$y = (Ax+B)e^{\beta x.}$$

Therefore for repeated roots β, we have

$$y = (Ax+B)e^{\beta x}.$$

Example 1.17. $y'' + 3y' + 2y = 0$.

$$\lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1) \qquad \alpha = -2, \beta = -1$$
$$\text{and } y = Ae^{-2x} + Be^{-x}.$$

Example 1.18. $y'' + 8y' + 16y = 0$.

$$\text{Here} \quad (\lambda+4)^2 = \lambda^2 + 8\lambda + 16 = 0,$$
$$\lambda = -4.$$

Therefore

$$y = (Ax+B)e^{-4x}.$$

Case when roots are complex. The roots of $\lambda^2 + m\lambda + n = 0$ may of course be complex numbers. The solution we have written down remains valid in the case where the roots are complex, but it is often convenient to express the solution using only real numbers. Recall that Euler's formula states

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Now if $\lambda = \alpha \pm i\beta$ are the roots of

$$\lambda^2 + m\lambda + n = 0,$$

then the general solution of

$$y'' + my' + ny = 0$$

will be

$$y = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}.$$

We use Euler's formula to rewrite this

$$\begin{aligned} y &= e^{\alpha t}\left(Ae^{i\beta t} + Be^{-i\beta t}\right) \\ &= e^{\alpha t}\left(A\left(\cos\beta t + i\sin\beta t\right) + B\left(\cos\beta t - i\sin\beta t\right)\right) \\ &= e^{\alpha t}\left[(A+B)\cos\beta t + i(A-B)\sin\beta t\right] \\ &= e^{\alpha t}\left(C_1\cos\beta t + C_2\sin\beta t\right), \end{aligned}$$

in which $C_1 = A + B$, $C_2 = i(A - B)$. Since A, B are allowed to be complex, then C_1 and C_2 may be real valued.

Example 1.19. Solve $y'' - 4y' + 13y = 0$.

$$\lambda^2 - 4\lambda + 13 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2}$$

$$= 2 \pm 3i$$

$$\alpha = 2, \quad \beta = 3.$$

So the general solution can be written

$$y = e^{2t} \left(C_1 \cos 3t + C_2 \sin 3t \right).$$

With complex roots, this is the preferred form of the solution.

1.10. **The Method of Undetermined Coefficients.** Imagine that we have a second order linear ODE with constant coefficients

$$y'' + ay' + by = f(x). \tag{1.28}$$

We will show later that solutions to (1.28) are of the form

$$y = y_h + y_p,$$

where y_h is the general solution of the homogeneous problem associated with (1.28),

$$y'' + ay' + by = 0,$$

and y_p is a *particular* solution to (1.28).

Variation of Parameters gives us a method of finding y_p knowing y_h. However, VOP can be quite complex and if $f(x)$ has a relatively simple form, this can be more trouble than it is worth.

Consider the problem where $f(x)$ is a polynomial $p(x)$. Clearly if we have a function y_p such that

$$y_p'' + ay_p' + by_p = p(x),$$

then y_p must itself be a polynomial of the same degree as $p(x)$. Thus we might pick an arbitrary polynomial $y_p(x)$ and try and choose coefficients so that y_p is a solution.

Example 1.20. $y'' + 2y' + 2y = x^2$.
Let us try a polynomial of degree 2.

$$y_p = ax^2 + bx + c.$$

Then if $y_p'' + 2y_p' + 2y_p = x^2$ we must have

$$2a + 2(2ax + b) + 2(ax^2 + bx + c) = x^2.$$

Now this implies that

$$\left.\begin{aligned} 2a &= 1, \\ 4a + 2b &= 0, \\ 2a + 2b + 2c &= 0, \end{aligned}\right\}$$

when we equate coefficients of powers of x. So $a = \frac{1}{2}$ and therefore

$$4 \cdot \frac{1}{2} + 2b = 0, \quad 2b = -2, \quad b = -1$$

$$\text{and} \quad 1 - 2 + 2c = 0,$$

$$\text{so } c = \frac{1}{2}.$$

Therefore we must have

$$y_p = \frac{1}{2}x^2 - x + \frac{1}{2}.$$

This is an example of the *method of undetermined coefficients.* Here is a useful table for the equation

$$y'' + ay' + by = R(x).$$

We look for y_p of the following forms

$R(x)$	Choice for y_p
$ke^{\gamma x}$	$ce^{\gamma x}$
$\sum_{k=0}^{n} a_k x^k$	$\sum_{k=0}^{n} b_k x^k$
$\alpha \cos wx + \beta \sin wx$	$A \cos wx + B \sin wx$
$e^{\alpha x}(\alpha \cos wx + \beta \sin wx)$	$e^{\alpha x}(A \cos wx + B \sin wx)$.

So if we want to solve, say

$$y'' + ay' + by = \alpha \cos wx + \beta \sin wx,$$

then we look for a particular solution of the form

$$y_p = A \cos wx + B \sin wx.$$

This technique, while limited in scope, entails less work than variation of parameters in the cases where it is applicable.

Example 1.21. Solve

$$y'' + 3y' + 2y = \sin x.$$

First solve

$$y'' + 3y' + 2y = 0.$$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0.$$

The roots are $\lambda = -2, \lambda = -1$ and therefore

$$y_h = Ae^{-2x} + Be^{-x}$$

Now $R(x) = \sin x$. So we try

$$y_p = C \cos x + D \sin x.$$

Then

$$y_p' = -C\sin x + D\cos x,$$
$$y_p'' = -C\cos x - D\sin x.$$

Hence

$$-C\cos x - D\sin x - 3C\sin x + 3D\cos x + 2C\cos x + 2D\sin x =$$
$$(-D - 3C + 2D)\sin x + (-C + 3D + 2C)\cos x = \sin x.$$

So

$$D - 3C = 1,$$
$$C + 3D = 0.$$

Hence $C = -3D$.

Thus

$$D - 3(-3D) = 10D = 1,$$

giving

$$D = \frac{1}{10},$$
$$C = -\frac{3}{10}.$$

So the particular solution is

$$y_p = -\frac{3}{10}\cos x + \frac{1}{10}\sin x,$$

and $y = y_h + y_p$ is the general solution.

Example 1.22. Solve

$$y'' + 5y' + 6y = 2e^x.$$

First solve

$$y'' + 5y' + 6y = 0.$$

The characteristic equation is

$$\lambda^2 + 5\lambda + 6 = 0,$$

which factorises as

$$(\lambda + 3)(\lambda + 2) = 0.$$

So the roots are $\lambda = -3, \lambda = -2$. and the homogeneous solution is $y_h = Ae^{-3x} + Be^{-2x}$.

$R(x) = 2e^x$. Now $2e^x$ is not a solution of the homogeneous problem, so we try

$$y_p = Ce^x.$$

Since

$$y_p'' + 5y_p' + 6y_p = 2e^x \quad \text{then}$$
$$(1+5+6)Ce^x = 12Ce^x = 2e^x,$$
$$\text{or} \quad C = \frac{1}{6}.$$

Thus

$$y_p = \frac{1}{6}e^x.$$

So our general solution is

$$y = y_h + y_p$$
$$y = Ae^{-3x} + Be^{-2x} + \frac{1}{6}e^x.$$

There is one slight modification to the above that sometimes has to be made.

We are trying to solve

$$y'' + ay' + by = R(x).$$

Now if

$$R''(x) + aR'(x) + bR(x) = 0$$

then we *cannot* have $y_p = cR(x)$ since

$$y_p'' + ay_p' + by_p = c(R'' + aR' + bR) = 0.$$

So what do we do?

Since our equation is constant coefficient, then solutions correspond to roots of

$$\lambda^2 + a\lambda + b = 0. \tag{1.29}$$

Rule 1. Let λ_1, λ_2 be roots of (1.29). Then if $R(x)$ is a solution it must have the form

$$R(x) = \alpha e^{\lambda_1 x} + \beta e^{\lambda_2 x}.$$

Rule 2. If $R(x)$ is a solution of the associated homogeneous problem, multiply the test solution in the table by

(i) x if $R(x)$ corresponds to a root of multiplicity 1,

(ii) x^2 if $R(x)$ corresponds to a root of multiplicity 2.

In this case we look for a y_p of the form

$$y_p = x\left(Ae^{\lambda_1 x} + Be^{\lambda_2 x}\right).$$

If $\lambda_{1,2}$ are complex with $\lambda_1 = \alpha + i\beta$, then R(x) must have the form

$$R = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right),$$

so we try

$$y_p = xe^{\alpha x}\left(C\cos\beta x + D\sin\beta x\right).$$

If $R(x)$ corresponds to a double root, then we multiply by x^2.

Let us see some examples of this procedure.

Example 1.23. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = xe^{-2x}$.
We first solve

$$y'' + 4y' + 4y = 0$$

leading to the characteristic equation $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$. Obviously $\lambda = -2$ is a root of multiplicity 2. So we look for a solution of the form

$$y_p = Cx^3e^{-2x} = x^2(Cxe^{-2x}).$$

Simple calculations give.

$$\begin{aligned}
\frac{dy_p}{dx} &= (3Cx^2 - 2Cx^3)e^{-2x}, \\
\frac{d^2y_p}{dx^2} &= 2(3Cx^2 - 2Cx^3)e^{-2x} + (6Cx - 6Cx^2)e^{-2x}.
\end{aligned}$$

Therefore

$$\begin{aligned}
y_p'' + 4y_p' + 4y_p &= \Big((4C - 8C + 4C)\,x^3 + (-12C + 12C)\,x^2 + 6Cx\Big)\,e^{-2x} \\
&= 6Cxe^{-2x} \\
&= xe^{-2x}
\end{aligned}$$

Hence $C = \frac{1}{6}$. Thus

$$y_p = \frac{1}{6}x^3e^{-2x}.$$

Example 1.24 (Resonance)**.** The situation in which R satisfies the associated homogeneous problem has real physical consequences. One is the phenomenon of resonance. Let us consider the ODE

$$y'' + 4y = \sin(2t), y(0) = 0, y'(0) = 1.$$

The solutions of $y'' + 4y = 0$ are $y_1 = \sin(2t)$, $y_2 = \cos(2t)$. Thus $R(t) = \sin(2t)$ satisfies the homogeneous problem. Let us try $y_p = At\sin(2t) + Bt\cos(2t)$. Then

$$y_p'' + 4y_p = 4A\cos(2t) - 4B\sin(2t) = \sin(2t).$$

So $A = 0, B = -1/4$. Consequently the general solution is

$$y = c_1\cos(2t) + c_2\sin(2t) - \frac{1}{4}t\cos(2t).$$

Now $y(0) = c_1 = 0$ and $y'(t) = 2c_2 \cos(2t) - \frac{1}{4}\cos(2t) + \frac{1}{2}t\sin(2t)$. Thus $y'(0) = 2c_2 - \frac{1}{4} = 1$. So $c_2 = \frac{5}{8}$. So the solution is

$$y(t) = \frac{1}{8}(5\sin(2t) - 2t\cos(2t)).$$

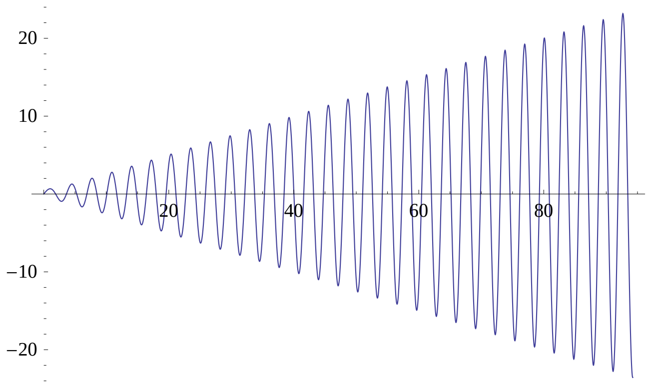

FIGURE 1. Resonance in a solution.

Let us now plot this solution. See Figure 1. Notice that the oscillations grow without bound. This phenomenon is called resonance. It is a real physical phenomenon which occurs when a system is subjected to a forcing term which matches the natural frequency of the system. Opera singers shattering wine glasses by hitting the right note provide an illustration of this. In the building of large structures, resonance needs to be taken into account to avoid instability in windy conditions.

2. Second Order Linear Differential Equations

We now embark upon a study of quite general second order ordinary differential equations of the form

$$y'' + p(x)y' + q(x)y = R(x). \tag{2.1}$$

This is the most general 2nd order linear ODE. It is nonhomogeneous, meaning that the term $R \neq 0$. The first question we need to consider is that of existence and uniqueness for solutions of (2.1).

Any nth order equation can be converted to a system of n first order equations. So let us first formulate Picard's Theorem for first order systems.That is, we want to know to study the system

$$\begin{aligned} y_1'(t) &= f_1(t, y_1, ..., y_n),\ y_1(0) = a_1 \\ &\vdots \\ y_n'(t) &= f_n(t, y_1, ..., y_n),\ y_n(0) = a_n. \end{aligned}$$

Equivalently we consider $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$, $\mathbf{y}(0) = \mathbf{a}$. We define the norm of a vector $\mathbf{x} = (x_1, ..., x_n)$ by

$$\|\mathbf{x}\| = \max_i\{|x_1|, ..., |x_n|\}.$$

A vector function $\mathbf{f}$ satisfies a Lipschitz condition if there is a constant K such that

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\| \leq K\|\mathbf{x} - \mathbf{y}\|, \tag{2.2}$$

for all $\mathbf{x}, \mathbf{y}$. Then we have the result

Theorem 2.1 (Picard's Theorem for Systems)**.** *If the continuous function* $\mathbf{f}$ *satisfies a Lipschitz condition on*

$$R = \{(t, \mathbf{x}) : |t - t_0| \leq a, \|\mathbf{x} - \mathbf{k}\| \leq b\},$$

and $\|f(t, \mathbf{x})\| \leq A$ *on* R *then the initial value problem*

$$\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t)),$$

$\mathbf{y}(t_0) = \mathbf{k}$ *has a unique solution on* $(t_0-\alpha, t_0+\alpha)$, *where* $\alpha = \min(a, b/A)$.

Proof. The proof is essentially the same as in the one dimensional case. We simply replace the absolute values in the one dimensional case with the norm and the details remain basically the same. ☐

Now we convert (2.1) to a first order system by setting $u = y, v = y'$ giving the system

$$u' = v \tag{2.3}$$

$$v' = -p(x)v - q(x)u + R(x), \tag{2.4}$$

$u(x_0) = y_0, v(x_0) = y_1$. Applying the system version of Picard's Theorem and imposing suitable assumptions on the functions p, q, R we can show that the equation $u'' + p(x)u' + q(x)u = R(x)$ with $u(x_0) = y_0$, $u'(x_0) = y_1$ has a unique solution over some interval.

Actually, we can do better than this with some work. We will not give the details, but simply state the most important result.

Theorem 2.2. *Let* (a, b) *be an interval on which* $p(x)$, $q(x)$, *and* $R(x)$ *are continuous. Let* $x_0 \in (a, b)$ *and let* $y_0, y_1 \in \mathbb{R}$, *and arbitrary. Then over the interval* (a, b) *equation (2.1) has one and only solution* $y = \phi(x)$ *satisfying*

$$y_0 = \phi(x_0), \quad y_1 = \phi'(x_0).$$

2.0.1. *Some important concepts.* Given an equation (2.1) the equation obtained by setting $R(x) \equiv 0$ is called the associated *homogenous* equation. We will consider homogeneous equations first.

Probably the most important property of (2.1) with $R(x) \equiv 0$ is the property of linearity.

Let

$$Ly = y'' + p(x)y' + q(x)y, \tag{2.5}$$

and let $Ly_1 = Ly_2 = 0$ and c_1, c_2 be constants. Then

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2 = 0.$$

That is if y_1 and y_2 are solutions of $Ly = 0$ then so is $c_1y_1 + c_2y_2$.

From linear algebra we recall the concept of *linear independence* of a set of vectors. A similar idea exists for functions.

Definition 2.3. A collection of n functions $\{y_1, ..., y_n\}$ on an interval (a, b) is linearly independent if and only if

$$c_1y_1(x) + \cdots + c_ny_n(x) = 0 \quad \text{for all } x \in (a, b),$$

if and only if $c_1 = \cdots = c_n = 0$.

If a set of functions is not linearly independent then it is said to be linearly dependent.

The meaning of this definition is easy to grasp. A set of functions is linearly independent if none of the functions in the set can be written as a linear combination of the others. If we have two functions, then they are linearly independent if and only if one is not a multiple of the other.

Example 2.1. e^x and e^{2x} are linearly independent. Whereas e^x, $2e^x$ are not linearly independent. The second is just twice the first. They are therefore linearly dependent.

Example 2.2. The functions $y_1(x) = x, y_2(x) = x^2, y_3(x) = 4x + 3x^2$ are linearly dependent because $y_3 = 4y_1 + 3y_2$.

The importance of this definition lies in the following result.

Theorem 2.4. *Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of*

$$Ly = 0. \tag{2.6}$$

Then every solution of (2.6) is of the form

$$c_1 y_1(x) + c_2 y_2(x).$$

Actually we can generalise this to higher order equations.

Theorem 2.5. *Suppose that the nth order linear ODE*

$$y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = 0$$

has n linearly independent solutions $y_1, ..., y_n$. Then every solution of the ODE is of the form

$$y = \sum_{k=1}^{n} c_k y_k,$$

$c_1, ..., c_n$ are constants.

Corollary 2.6. *A linear ordinary differential equation of order n can have at most n linearly independent solutions.*

This leads to an important idea.

Definition 2.7. Given two linearly independent solutions y_1, y_2 of (2.1), the function $y = c_1 y_1(x) + c_2 y_2(x)$ for arbitrary c_1, c_2 is called the *general solution* of (2.6).

We will prove Theorem 2.4 shortly. The proof in the nth order case is basically the same. To this end, we introduce an important quantity in the theory of differential equations. Namely the Wronskian.

Definition 2.8. Let $y_1(x)$ and $y_2(x)$ be any two solutions of

$$y'' + p(x)y' + q(x)y = 0.$$

Then we define the *Wronskian* of $y_1(x)$ and $y_2(x)$ by

$$\begin{aligned} W(y_1, y_2) &= y_1(x) y_2'(x) - y_2(x) y_1'(x) \\ &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \end{aligned}$$

More generally if $\{y_1, ..., y_n\}$ are a set of $n-1$ times differentiable functions, then

$$W(y_1, ..., y_n) = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}. \tag{2.7}$$

The Wronskian tells us whether or not a set of functions are linearly independent.

Theorem 2.9. *Let $y_1, ..., y_n$ be defined and possess $n-1$ derivatives on an interval I. Then $y_1, ..., y_n$ are linearly independent on I, if and only if their Wronskian is nonzero for at least one point $x \in I$.*

Remark 2.10. If the Wronskian is nonzero at a point for a family of continuous functions, it will be nonzero in an interval around that point. We will not prove this.

Proof. This is an exercise in linear algebra. Assume that our functions are linearly dependent. Then for some non zero constants $a_1, ..., a_n$ we can write

$$a_1 y_1(x) + a_2 y_2(x) + \cdots + a_n y_n(x) = 0,$$

for all $x \in I$. Differentiating this $n-1$ times, we still get zero. Hence

$$\begin{aligned} a_1 y_1(x) + a_2 y_2(x) + \cdots + a_n y_n(x) &= 0, \\ a_1 y_1'(x) + a_2 y_2'(x) + \cdots + a_n y_n'(x) &= 0, \\ &\vdots \\ a_1 y_1^{(n-1)}(x) + a_2 y_2^{(n-1)}(x) + \cdots + a_n y_n^{(n-1)}(x) &= 0. \end{aligned}$$

This system of equations for $a_1, ..., a_n$ has a nonzero solution valid for all $x \in I$. Since $a_1 = a_2 = \cdots = a_n = 0$ is also a solution, the system does not have a unique solution. Thus the determinant of the system must equal zero for all $x \in I$. But the determinant is the Wronskian. So if the functions $y_1, ..., y_n$ are linearly dependent, $W(y_1, ..., y_n) = 0$ for every $x \in I$. Hence if the functions are linearly independent the Wronskian must be nonzero at at least one $x \in I$. ☐

Theorem 2.11. *If y_1, y_2 satisfy*

$$y'' + p(x)y' + q(x)y = 0,$$

then

$$W(y_1, y_2) = K_{12} e^{-\int p(x)dx},$$

K_{12} is a constant depending on y_1 and y_2.

Proof. y_1, y_2 satisfy (2.6). So

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \quad \text{(A)}$$
$$y_2'' + p(x)y_2' + q(x)y_2 = 0. \quad \text{(B)}$$

Consider the difference

$$\begin{aligned} y_1(B) - y_2(A) & \\ &= y_1(y_2'' + p(x)y_2' + q(x)y_2) - y_2(y_1'' + p(x)y_1' + q(x)y_1) \\ &= y_1 y_2'' - y_2 y_1'' + y_1 y_2' p(x) - y_2 y_1' p(x) + (y_1 y_2 - y_2 y_1) q(x) \\ &= y_1 y_2'' - y_2 y_1'' + (y_1 y_2' - y_2 y_1') p(x) = 0. \end{aligned}$$

Now

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_2 y_1' \\ \frac{d}{dx} W(y_1, y_2) &= y_1 y_2'' + y_2' y_1' - y_2' y_1' - y_2 y_1'' \\ &= y_1 y_2'' - y_2 y_1''. \end{aligned}$$

So

$$\frac{dW}{dx} + p(x)W = 0.$$

Solving this separable ODE gives

$$\frac{dW}{W} = -p(x)dx.$$

Hence

$$\ln W = -\int p(x)dx \quad \text{or}$$

$$W(y_1, y_2) = K_{12} e^{-\int p(x)dx}.$$

K_{12} depends upon $W(y_1(x_0), y_2(x_0))$ for some $x_0 \in (a, b)$. □

Given any two linearly independent functions y_1 and y_2 which can be differentiated, then $W(y_1, y_2) \neq 0$. We now recast Theorem 2.5 in a slightly different form.

Theorem 2.12. *If y_1, y_2 are solutions of (2.6) on (a, b) and*

$$W(y_1, y_2) \neq 0,$$

for some $x_0 \in (a, b)$, then any solution y of (2.6) may be written

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Proof. Let y_3 be any solution of (2.1). Then by Theorem 2.11

$$\left.\begin{aligned} W(y_3, y_1) &= y_3 y_1' - y_3' y_1 = K_{13} e^{-\int p(x)dx} \\ W(y_3, y_2) &= y_3 y_2' - y_3' y_2 = K_{23} e^{-\int p(x)dx} \end{aligned}\right\}.$$

This is a pair of simultaneous equations for y_3 and y_3'. We get by simple linear algebra

$$y_3(y_1 y_2' - y_2 y_1') = K_{23} e^{-\int p(x)dx} y_1(x) - K_{13} e^{-\int p(x)dx} y_2(x).$$

So

$$y_3 K_{12} e^{-\int p(x)dx} = K_{23} e^{-\int p(x)dx} y_1(x) - K_{13} e^{-\int p(x)dx} y_2(x),$$

and $K_{12} \neq 0$. Thus we have

$$y_3 = \frac{K_{23}}{K_{12}} y_1(x) - \frac{K_{13}}{K_{12}} y_2(x),$$

which we can write as

$$y_3 = c_1 y_1(x) + c_2 y_2(x).$$

□

Remark 2.13. This result extends to higher order equations. If

$$y^{(n)}(x) + P_{n-1}(x)y^{(n-1)} + \ldots + P_n y = 0, \tag{2.8}$$

and $y_1, \ldots, y_n$ are n linearly independent solutions of (2.8), then any solution y of (2.8) may be written

$$y(x) = \sum_{k=1}^{n} c_k y_k(x).$$

Example 2.3. Consider the equation

$$y'' - y = 0. \tag{2.9}$$

We see that $y_1 = e^x$, $y_2 = e^{-x}$ are solutions, so that

$$\begin{aligned} W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} \\ &= -e^x(e^{-x}) - e^x e^{-x} \\ &= -2 \neq 0. \end{aligned}$$

Therefor any solution of (2.9) is of the form

$$y = c_1 e^x + c_2 e^{-x}.$$

Example 2.4. An equation with trigonometric solutions is

$$y'' + 4y = 0. \tag{2.10}$$

One easily finds that $y_1 = \sin 2x$, $y_2 = \cos 2x$ are solutions and

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} \\ &= 2(\sin^2 2x + \cos^2 2x) \\ &= -2 \neq 0. \end{aligned}$$

Therefore any solution of (2.10) may be written

$$y = c_1 \sin 2x + c_2 \cos 2x.$$

The Wronskian is useful for the study of many different aspects of DEs. We give one important application. That of finding a second solution of an equation given a first solution.

Let us imagine that we have a solution y_1 of an equation

$$y'' + p(x)y' + q(x)y = 0.$$

We can often find such solutions by inspection. For example, an equation of the form

$$a(x)y''(x) + kxy'(x) - ky(x) = 0,$$

obviously has $y(x) = x$ as a solution. Can we find another linearly independent solution?

Proposition 2.14. *Let y_1 be a nonzero solution of*

$$y''(x) + p(x)y' + q(x)y = 0.$$

Then

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1(x)^2} dx, \tag{2.11}$$

is a second linearly independent solution.

Proof. We use the Wronskian $W(y_1, y_2)$. By our previous results we have for any second linearly independent solution y_2,

$$W(y_1, y_2) = Ae^{-\int p(x)dx}, \quad \text{for some constant } A \neq 0.$$

Now we differentiate the quantity $\frac{y_2}{y_1}$. By the quotient rule,

$$\begin{aligned} \frac{d}{dx}\left(\frac{y_2}{y_1}\right) &= \frac{y_1 y_2' - y_2 y_1'}{y_1^2} \\ &= \frac{W(y_1, y_2)}{y_1^2} \\ &= \frac{Ae^{-\int p(x)dx}}{y_1^2}. \end{aligned}$$

Thus

$$\frac{y_2}{y_1} = A \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

$$\text{So } y_2 = Ay_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2} dx,$$

is our second solution. □

Corollary 2.15. *Every second order linear ODE with one nonzero solution has exactly two linearly independent solutions.*

Example 2.5. Consider the equation

$$x^2 y'' + 4xy' - 4y = 0. \tag{2.12}$$

An equation of the form $x^2y'' + bxy' + cy = 0$ is known as an Euler equation and it may be solved by finding solutions of the form $y = x^\lambda$. Substitution of this into our equation will give us a quadratic in λ. However, as we remarked above, it it is easy to see by inspection that since $b = -c$, then $y = x$ is a solution. Thus by Proposition 2.14 we may find a second solution. First we rewrite the equation as

$$y'' + \frac{4}{x}y' - \frac{4}{x^2}y = 0.$$

This puts it into the form given in the proposition. Then $p(x) = \frac{4}{x}$. Now

$$e^{-\int p(x)dx} = e^{-4\int \frac{1}{x}dx} = e^{-4\ln x} = x^{-4}.$$

So

$$\begin{aligned} y_2(x) = Ay_1(x)\int \frac{\frac{1}{x^4}}{x^2}dx &= Ax\int \frac{dx}{x^6} \\ &= Ax\left(\frac{-1}{5x^5}\right) \\ &= -\frac{A}{5x^4}. \end{aligned}$$

Since any A is linear, any multiple of a solution is a solution. Hence x^{-4} is a solution. So we conclude that the general solution of (2.12) is

$$y = c_1x + c_2x^{-4}.$$

Example 2.6. A harder example follows. Suppose that we want to solve

$$u'' + xu' + u = 0.$$

It is not hard to see that $u_1(x) = e^{-\frac{x^2}{2}}$ is a solution since

$$\begin{aligned} u_1'(x) &= -xe^{-\frac{x^2}{2}} \\ u_1''(x) &= x^2e^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}} \\ &= (x^2-1)e^{-\frac{x^2}{2}}. \end{aligned}$$

So

$$(x^2-1)e^{-\frac{x^2}{2}} + x\left(-xe^{-\frac{x^2}{2}}\right) + e^{-\frac{x^2}{2}} = 0.$$

Now $p(x) = x$. So $e^{-\int p(x)dx} = e^{-\frac{x^2}{2}}$.

$$u_1^2(x) = \left(e^{-\frac{x^2}{2}}\right)^2 = e^{-\frac{x^2}{2}}.$$

So our second solution $u_2(x)$ is

$$\begin{aligned} u_2(x) &= Ae^{-\frac{x^2}{2}}\int \frac{e^{-\frac{x^2}{2}}}{e^{-x^2}}dx \\ &= Ae^{-\frac{x^2}{2}}\int e^{\frac{x^2}{2}}dx. \end{aligned}$$

We cannot do the integral so we leave the answer as

$$u_2(x) = Ae^{-\frac{x^2}{2}}\int_0^x e^{\frac{t^2}{2}}dt + u_2(0).$$

So any solution of $u'' + xu' + u = 0$ is of the form

$$u(x) = c_1e^{-\frac{x^2}{2}} + c_2\left(e^{-\frac{x^2}{2}}\int_0^x e^{\frac{t^2}{2}}dt + u_2(0)\right).$$

Example 2.7. Observe that

$$y'' + 3y' + 2y = 0. \tag{2.13}$$

has a solution $y = e^{-x}$. In this case $P(x) = 3$. So

$$e^{-\int 3dx} = e^{-3x}.$$

$y_1^2(x) = e^{-2x}$. So

$$\int \frac{e^{-\int P(x)dx}}{y_1^2(x)} = \int e^{-x}dx = -e^{-x}.$$

Hence

$$y_2(x) = Ae^{-x}(-e^{-x}) = -Ae^{-2x}.$$

Thus $y_2(x) = e^{-2x}$ is a solution and every solution of (2.13) is of form

$$y = c_1e^{-2x} + c_2e^{-x}.$$

Example 2.8. Suppose that we wish to solve $y'' + 2ay' + a^2y = 0$. We let $y = e^{\lambda x}$ and this gives the auxilliary equation

$$\lambda^2 + 2a\lambda + a^2 = 0$$

Then $p(x) = 2a$. So that with $y_1 = e^{-ax}$ we have

$$y_2 = y_1 \int \frac{e^{-2ax}}{e^{-2ax}}dx = xe^{-ax}. \tag{2.14}$$

This is the second solution that we arrived at previously.

2.1. **Non-Homogeneous Equations.** We now return to the problem of solving the inhomogeneous equation

$$Ly = y'' + p(x)y' + q(x)y = R(x). \tag{2.15}$$

The basic result we need is one telling us what solutions of nonhomogenous equations look like.

Theorem 2.16. *Let y_p be a particular solution of (2.15) and $y_h = c_1y_1 + c_2y_2$ be the general solution of the associated homogeneous problem*

$$y'' + p(x)y' + q(x)y = 0. \tag{2.16}$$

Then every solution of (2.15) is of the form

$$y = y_h + y_p.$$

Proof. Let y be any solution of (2.15) and y_p be any particular solution.

Now set

$$u = y - y_p.$$

Then

$$\begin{aligned} Lu &= L(y - y_p) \\ &= Ly - Ly_p \\ &= R(x) - R(x) \\ &= 0. \end{aligned}$$

Thus u is a solution of the associated homogeneous equation. Hence we can write

$$u = c_1 y_1(x) + c_2 y_2(x),$$

where y_1, y_2 are the two linearly independent solutions of the homogeneous problem. Thus our arbitrary solution y satisfies

$$c_1 y_1 + c_2 y_2 = y - y_p.$$

So

$$y = c_1 y_1 + c_2 y_2 + y_p.$$

□

2.2. **The Method of Variation of Parameters.** Given an equation

$$y'' + p(x)y' + q(x)y = R(x), \tag{2.17}$$

can we construct a particular solution $y_p(x)$? It turns out that given two linearly independent solutions y_1, y_2 of the associated homogeneous problem (2.17), we can produce a solution of (2.17) by a process called *variation of parameters.*

The idea is relatively straightforward. Let y_1, y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

with $W(y_1, y_2) = y_1 y_2' - y_2 y_1' \neq 0$.

Then we look for a solution of (2.17) of the form

$$y_p = u(x)y_1(x) + v(x)y_2(x).$$

We can certainly write any nonzero function y_p in this form. The question is whether we can determine u and v?

The key is that y_p must satisfy the differential equation, so we differentiate both sides.

$$y_p' = u'y_1 + uy_1' + vy_2' + v'y_2$$

and

$$\begin{aligned} y_p'' &= u''y_1 + u'y_1' + u'y_1' + uy_1'' + v''y_2 + v'y_2' + v'y_2' + vy_2'' \\ &= u''y_1 + 2u'y_1' + uy_1'' + v''y_2 + 2v'y_2' + vy_2''. \end{aligned}$$

Next we substitute into (2.17). We have

$$u''y_1 + 2u'y_1' + uy_1'' + v''y_2 + 2v'y_2' + vy_2'' +$$
$$p(x)(u'y_1 + uy_1' + vy_2' + v'y_2) + q(x)(u(x)y_1(x) + v(x)y_2(x)) = R(x).$$

Now we collect the terms together and we get

$$u(y_1'' + p(x)y_1' + q(x)y_1) + v(y_2'' + p(x)y_2' + q(x)y_2) +$$
$$u''y_1 + v''y_2 + 2(u'y_1' + v'y_2') + p(x)(u'y_1 + v'y_2) = R(x),$$

where $y_1'' + p(x)y_1' + q(x)y_1 = A$ and $y_2'' + p(x)y_2' + q(x)y_2 = B$. Notice that A and B are zero since y_1, y_2 are solutions of the homogeneous problem. So that

$$u''y_1 + v''y_2 + 2(u'y_1' + v'y_2') + p(x)(u'y_1 + v'y_2) = R(x). \tag{2.18}$$

The idea here is to choose u and v in such a way that

$$u'y_1 + v'y_2 = 0,$$

which means that $u''y_1 + v''y_2 + 2(u'y_1' + v'y_2') = R(x)$ Now observe that

$$\frac{d}{dx}(u'y_1 + v'y_2) = u''y_1 + v''y_2 + u'y_1' + v'y_2' = 0.$$

Substituting this into (2.18) to obtain the second equation

$$u'y_1' + v'y_2' = R(x).$$

So we have a simultaneous pair of equations for u', v'. These are

$$u'y_1 + v'y_2 = 0 \qquad \text{(A)}$$
$$u'y_1' + v'y_2' = R(x). \qquad \text{(B)}$$

To solve, multiply A by y_2' and B by y_2. So

$$u'y_1y_2' + v'y_2y_2' = 0, \qquad \text{A}',$$
$$u'y_1'y_2 + v'y_2'y_2 = R(x)y_2. \qquad \text{B}'.$$

Thus $A' - B'$ gives

$$u'(y_1y_2' - y_1'y_2) = -y_2R(x),$$
$$\text{or} \quad u' = \frac{-y_2R(x)}{y_1y_2' - y_1'y_2} = \frac{-y_2R(x)}{W(y_1, y_2)}$$
$$= \frac{-y_2R(x)}{K_{12}e^{-\int p(x)dx}}.$$

Which gives

$$u' = \frac{-y_2R(x)}{K_{12}}e^{\int p(x)dx}.$$

For v' we obtain

$$v' = \frac{y_1R(x)}{W(y_1, y_2)}$$
$$= \frac{y_1R(x)}{K_{12}}e^{\int p(x)dx}.$$

Integrating our equations for u' and v' we obtain

$$u = -\int \frac{y_2 R(x)}{W(y_1, y_2)} dx,$$

and

$$v = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx.$$

Finally $y_p = uy_1 + vy_2$. We differentiate this and substitute it into the original ODE to show that it is a solution. This is a somewhat laborious exercise and we omit it. We have derived the following result.

Theorem 2.17. *Let y_1 and y_2 be linearly solutions of the second order linear equation $y'' + p(x)y' + q(x)y = 0$. We suppose that p and q are smooth functions and the ODE has a solution. Then if*

$$u = -\int \frac{y_2 R(x)}{W(y_1, y_2)} dx$$
$$v = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx,$$

the function $y_p = uy_1 + vy_2$ is a solution of the equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = R(x).$$

Now we will consider some examples.

Example 2.9.

$$x^2 y'' + 4xy' - 4y = x^2. \tag{2.19}$$

We saw earlier that $y_1 = x$ and $y_2 = x^{-4}$ are solutions of the homogeneous problem.

First we compute the Wronskian.

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_2 y_1' \\ &= x\left(-\frac{4}{x^5}\right) - \frac{1}{x^4} \\ &= -\frac{5}{x^4}. \end{aligned}$$

We see that the differential equation is

$$y'' + \frac{4}{x} y' - \frac{4}{x^2} y = 1. \tag{2.20}$$

Thus $R(x) = 1$. Consequently

$$u = -\int \frac{(x^{-4})}{-5/x^4} dx = \frac{1}{5}\int dx = \frac{1}{5}x,$$
$$v = \int \frac{x}{-5/x^4} dx = -\frac{1}{5}\int x^5 dx = -\frac{1}{30}x^6.$$

Therefore the solution y_p is

$$y_p = \frac{1}{5}x \cdot x - \frac{1}{30}x^6 \cdot \frac{1}{x^4} = \left(\frac{1}{5} - \frac{1}{30}\right)x^2 = \frac{1}{6}x^2.$$

Example 2.10. Let us solve the ODE

$$x^2y'' + xy' - 4y = \ln x.$$

$y_1 = x^2$ is a solution of the homogeneous problem.

We now find a second solution y_2. We have the equation

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2}.$$

$p(x) = \frac{1}{x}$, so $e^{-\int p(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln\frac{1}{x}} = \frac{1}{x}$.

Then our second linearly independent solution y_2 is

$$\begin{aligned} y_2 = y_1 \int \left(\frac{1}{x}/x^4\right) dx &= x^2 \int \frac{dx}{x^5} \\ &= x^2\left(-\frac{1}{4}x^{-4}\right) \\ &= -\frac{1}{4}x^{-2}. \end{aligned}$$

So we take $y_2 = x^{-2}$.

Now we find y_p by variation of parameters. So

$$u = -\int \frac{y_2(x)R(x)}{W(y_1,y_2)}dx, \quad v = \int \frac{y_1(x)R(x)}{W(y_1,y_2)}dx,$$

$$\text{and} \quad y = uy_1 + vy_2.$$

Calculating the Wronskian

$$\begin{aligned} W(y_1,y_2) = y_1y_2' - y_2y_1' &= x^2(-2x^{-3}) - 2x \cdot x^{-2} \\ &= -2x^{-1} - 2x^{-1} = -4x^{-1} = -\frac{4}{x}. \end{aligned}$$

Thus

$$\begin{aligned} u &= -\int \frac{x^{-2}\frac{\ln x}{x^2}}{-\frac{4}{x}}dx \\ &= \frac{1}{4}\int \frac{\ln x}{x^3}dx \\ &= \frac{1}{4}\left(-\frac{1}{2x^2}\ln x + \int \frac{1}{x} \cdot \frac{1}{2x^2}dx\right) \\ &= \frac{1}{4}\left(-\frac{\ln x}{2x^2} + \frac{1}{2}\left(-\frac{1}{2}x^{-2}\right)\right) \\ &= -\frac{\ln x}{8x^2} - \frac{1}{16} \cdot \frac{1}{x^2} \end{aligned}$$

and

$$\begin{aligned} v &= \int \left(\frac{x^2 \frac{\ln x}{x^2}}{-4/x} \right) dx \\ &= -\frac{1}{4} \int x \ln x dx \\ &= -\frac{1}{4} \left(\frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \right) \\ &= \frac{1}{16} x^2 (1 - 2 \ln x) \end{aligned}$$

Thus

$$\begin{aligned} y_p &= x^2 \left(-\frac{\ln x}{8x^2} - \frac{1}{16x^2} \right) + \frac{1}{x^2} \left(-\frac{x^2}{8} \ln x + \frac{x^2}{16} \right) \\ &= -\frac{\ln x}{8} - \frac{1}{16} - \frac{\ln x}{8} + \frac{1}{16} \\ &= -\frac{\ln x}{4}. \end{aligned}$$

So the general solution is

$$y = c_1 x^2 + c_2 x^{-2} - \frac{\ln x}{4}.$$

Example 2.11.

$$y'' + y = \sec x \tag{2.21}$$

First solve $y'' + y = 0$. We have the auxiliary equation $\lambda^2 + 1 = 0$, which has solutions $\lambda = \pm i$. Thus

$$y = A \sin x + B \cos x.$$

Take

$$\begin{aligned} y_1 &= \sin x, \\ y_2 &= \cos x. \end{aligned}$$

Now

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_2 y_1' \\ &= \sin x(-\sin x) - \cos x \cos x \\ &= -1. \end{aligned}$$

Then

$$
\begin{aligned}
u &= \int \frac{\cos x \sec x}{-1} dx \\
&= \int dx \\
&= x. \\
v &= \int \frac{\sin x \sec x}{-1} dx \\
&= -\int \tan x dx \\
&= -\ln|\sec x| \\
&= \ln|\cos x|.
\end{aligned}
$$

Hence

$$y_p = x \sin x + \cos x \ln|\cos x|.$$

3. Series Solutions of ODEs

3.1. **Introductory Examples.** In the previous section we discussed the problem of solving some ODEs. One of the problems that we looked at was the problem of solving second order, linear constant coefficient equations.

Given an ODE with real, constant coefficients

$$ay''(x) + by'(x) + cy(x) = 0, \tag{3.1}$$

we know that it can be solved by looking for solutions of the form $y(x) = e^{\lambda x}$. This leads to a quadratic equation for λ and the various cases which we considered previously.

This can be extended to higher order linear equations as well. For example, suppose we wish to solve

$$y''' + 5y'' - 2y' - 24y = 0. \tag{3.2}$$

Letting $y = e^{\lambda x}$ gives $\lambda^3 + 5\lambda^2 - 2\lambda - 24 = 0$. The roots are $2, -3, -4$ and so the general solution is

$$y = c_1e^{2x} + c_2e^{-3x} + c_3e^{-4x}.$$

In practice for higher order equations, we may not always be able to solve the polynomial equation for the characteristic values $\lambda_1, ..., \lambda_n$ exactly, but we can always employ some numerical root finding scheme. So linear constant coefficient ODEs present no challenge in terms of obtaining solutions.

An obvious question to ask is what happens when we allow the equations we are studying to have coefficients which are not constant?

The simplest ODEs which are not constant coefficient equations have powers of x as coefficients of the derivatives of y. The easiest such equation to solve is of the form

$$ax^2y'' + bxy' + cy = 0. \tag{3.3}$$

We have seen these Euler type equations already. They are easy to solve since we can use the same method that works for (3.1). Instead of looking for an exponential solution, we look for a power of x as a solution. That is, we try $y = x^\lambda$ as our trial solution.

Differentiation gives $y' = \lambda x^{\lambda-1}$ and $y'' = \lambda(\lambda-1)x^{\lambda-2}$. Substitution into the ODE (3.3) gives

$$a\lambda(l-1)x^2x^{\lambda-2} + bx\lambda x^{\lambda-1} + cx^\lambda = x^\lambda(a\lambda^2 + (b-a)\lambda + c) = 0.$$

Cancelling the x^λ term gives us the quadratic

$$a\lambda^2 + (b-a)\lambda + c = 0.$$

Example 3.1. Solve the ODE $x^2y'' + 5xy' + 3y = 0$.
Solution We set $y = x^\lambda$. Substitution into the ODE leads to the quadratic equation

$$\lambda^2 + 4\lambda + 3 = 0.$$

This has roots -3 and -1. Thus the general solution of the ODE is

$$y = \frac{A}{x} + \frac{B}{x^3}.$$

By means of this method, any ODE of the form (3.3) may be solved. Does this work for other kinds of equations where the pattern of the powers of x does not match the order of the derivatives as it does here? An example would be the equation $y'' - xy = 0$. Here the coefficient of the zeroth derivative is 2 and the coefficient of the second derivative is zero. We can easily check that this does not have solutions of the form $y = x^\lambda$ for any λ.

Can we solve such an equation? Actually we will shortly see that the general solution of $y'' - xy = 0$ is

$$y = c_1 Ai(x) + c_2 Bi(x)$$

where Ai and Bi are the so called *Airy* functions of the first and second kind. Airy functions are examples of what are known as *special functions*. Special functions have been the subject of intense study by mathematicians since the late 18th century. The exact definition of a special function is somewhat difficult to formulate. For our purposes, a special function is one which is not of the elementary type familiar from high school. i.e $\ln x, e^x, \sin x$ are all elementary functions. Special functions are often (though not always) solutions of second order ODES.

The key to solving non-constant coefficient equations is to look for solutions which are given in terms of infinite series. We take an infinite power series

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

and substitute it into the differential equation and obtain a formula for the coefficients a_n. The linear change of variable $z = x - x_0$ will convert this series to one of the form $y = \sum_{n=0}^{\infty} a_n z^n$. So we will focus our attention on power series solutions with $x_0 = 0$.

The technique of series solutions relies upon some important results from real analysis, which we now state without proof.

Theorem 3.1. *Suppose that the power series* $\sum_{n=0}^{\infty} a_n x^n$ *is convergent on the interval* $(-R, R)$, *for* $R > 0$. *Then* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is differentiable on* $(-R, R)$ *and*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \tag{3.4}$$

for all $x \in (-R, R)$.

It follows from this that every power series is differentiable infinitely many times, and further, the radius of convergence of the power series is not changed by the process of differentiation.

Using power series to solve differential equations requires that we equate a power series to zero. We then need to know under what conditions a power series is equal to zero for all x?

Proposition 3.2. *Suppose that*

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all $x \in (-R, R)$, where $R > 0$ is the radius of convergence of both series. Then for each n, we have $a_n = b_n$.

Corollary 3.3. *If $\sum_{n=0}^{\infty} a_n x^n = 0$ for all $x \in (-R, R)$, where $R > 0$ is the radius of convergence of the power series, then $a_n = 0$ for every $n = 0, 1, 2, 3, ...$*

We are now in a position to develop series methods for the solution of ordinary differential equations. A simple example will illustrate the basic ideas.

Example 3.2. Obtain a series solution of the ODE

$$y' = y \tag{3.5}$$

with $y(0) = 1$.
Solution We want to try an infinite series solution. That is, we want a trial solution with infinitely many terms:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + \quad \dots \quad + a_n x^n + \dots \\ \text{so } y' &= \quad a_1 \quad + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + (n+1) a_{n+1} x^n \\ &\quad + \dots \end{aligned}$$

The differential equation (3.5) implies that for every value of x we must have

$$a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + a_n x^n + \dots \tag{3.6}$$
$$= a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + (n+1) a_{n+1} x^n + \dots \tag{3.7}$$

In order for the power series to solve the ODE for every value of x the coefficients of the powers of x on both sides of (3.6) must match. It is easy to see that we have to have $a_0 = a_2$, $a_1 = 2a_2$ etc. Putting this in a tabular form gives

$$
\begin{cases}
a_1 = a_0 = \frac{a_0}{1} \\
a_2 = \frac{1}{2}a_1 = \frac{a_0}{1\times 2} \\
a_3 = \frac{1}{3}a_2 = \frac{1}{1\times 2\times 3}a_0 = \frac{1}{3!}a_0 \\
\quad\vdots \\
a_n = \frac{1}{n}a_{n-1} = \frac{1}{n!}a_0 \\
a_{n+1} = \frac{1}{n+1}a_n = \frac{1}{(n+1)!}a_0 \\
\quad\vdots
\end{cases}
\tag{3.8}
$$

This process of courses continues forever for all coefficients. The point is that we have a relationship between the coefficients. This allows us to obtain all coefficients in terms of a_0. The initial condition $y(0) = 1$ now implies

$$1 = y(0) = a_0.$$

Consequently we have $a_n = 1/n!$ for every n.

Hence we have obtained a solution to our equation in terms of an infinite series

$$
\begin{aligned}
y &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \dots \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!}.
\end{aligned}
\tag{3.9}
$$

We already know that the ODE $y' = y$ with $y(0) = 1$ has the unique solution $y = e^x$. So we expect that this series is simply the Taylor series for e^x, and this is the case. If we had no initial condition, the constant a_0 would remain arbitrary and the solution would be given as $y = a_0 \sum_{n=0}^{\infty} x^n/n!$.

Let us consider another familiar ODE from the point of view of power series.

Example 3.3. Use series methods to solve the ODE

$$y'' + \omega^2 y = 0.$$

Solution Again we try a power series solution $y = \sum_{n=0}^{\infty} a_n x^n$. Differentiation produces the following.

$$
\begin{aligned}
y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.
\end{aligned}
$$

We now substitute this into the ODE to get

$$y'' + \omega^2 y = 0.$$

This becomes

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \omega^2 \sum_{n=0}^{\infty} a_n x^n = 0. \tag{3.10}$$

We now make the powers of x in the sums the same. Notice that the first sum in (3.10) starts at $n = 2$. However the power of x is $n - 2$. So the first term in this series will be x^0, the next will be x^1 etc. So by replacing n by $n + 2$ in the first sum, the expression $n(n - 1)$ becomes $(n + 2)(n + 1)$ and so the equation (3.10) can be written

$$\sum_{n+2=2}^{\infty} (n+2)(n+1)a_{n+2} x^n + \omega^2 \sum_{n=0}^{\infty} a_n x^n = 0,$$

which is the same as

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \omega^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

The key here is that we have made the sums both start from $n = 0$. We next combine the sums and in each term take out the common power of x to obtain the expression

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + \omega^2 a_n\right] x^n = 0.$$

This has to hold for all values of x. Which means that the coefficients of x must be zero.

We therefore equate the coefficients of like powers of x to zero, which gives us

$$(n+2)(n+1)a_{n+2} + \omega^2 a_n = 0$$

for $n = 0, 1, 2, \ldots$

What we have here is a recurrence relation for the coefficients a_n. It is simply

$$a_{n+2} = \frac{-\omega^2}{(n+2)(n+1)} a_n$$

for $n = 0, 1, 2, \ldots$

The next step is to identify coefficients recursively. We will find that they naturally separate into the odd and even values of n. Taking $n = 0$ gives

$$a_2 = -\frac{\omega^2}{2 \times 1} a_0 = -\frac{\omega^2}{2!} a_0 \tag{3.11}$$

$$a_4 = -\frac{\omega^2}{4 \times 3} a_2 = \frac{(-1)^2 \omega^4}{4!} a_0 \tag{3.12}$$

$$a_6 = -\frac{\omega^2}{6 \times 5} a_4 = \frac{(-1)^3 \omega^6}{6!} a_0. \tag{3.13}$$

If we take $n = 1$ we get

$$a_3 = -\frac{\omega^2}{3 \times 2} a_1 = -\frac{\omega^2}{3!} a_1 \tag{3.14}$$

$$a_5 = -\frac{\omega^2}{5 \times 4} a_3 = \frac{(-1)^2 \omega^4}{5!} a_1 \tag{3.15}$$

$$a_7 = -\frac{\omega^2}{7 \times 6} a_5 = \frac{(-1)^3 \omega^6}{7!} a_1, \tag{3.16}$$

and in general the odd and even coefficients are given by

$$a_{2n} = \frac{(-1)^n \omega^{2n}}{(2n)!} a_0, \qquad a_{2n+1} = \frac{(-1)^n \omega^{2n}}{(2n+1)!} a_1$$

respectively.

From this we can now determine the general solution of our ODE. We have

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + \dots \\ &= a_0 + a_1 x - a_0 \frac{\omega^2}{2!} x^2 - a_1 \frac{\omega^2}{3!} x^3 + - \dots \\ &= a_0 \left(1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} \dots + \frac{(-1)^n (\omega x)^{2n}}{(2n)!} + \dots \right) \\ &\quad + \frac{a_1}{\omega} \left((\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} \dots + \frac{(-1)^n (\omega x)^{2n+1}}{(2n+1)!} + \dots \right). \end{aligned}$$

Let us introduce some initial conditions to tidy up the solution. We will take $y(0) = y_0$ and $y'(0) = y_1$. This implies that

$$y_0 = y(0) = a_0(1 + 0 + \dots) + \frac{a_1}{\omega}(0 + 0 + \dots).$$

So $a_0 = y_0$. Next we have

$$y_1 = y'(0) = a_0(0 + 0 + \dots) + \frac{a_1}{\omega}(\omega + 0 + \dots),$$

which gives $a_1 = y_1$.

It is now an easy task to find the particular solution that satisfies the initial conditions:

$$y(x) = y_0 \left(\sum_{n=0}^{\infty} \frac{(-1)^n (\omega x)^{2n}}{(2n)!} \right) + \frac{y_1}{\omega} \left(\sum_{n=0}^{\infty} \frac{(-1)^n (\omega x)^{2n+1}}{(2n+1)!} \right).$$

These solutions are not so mysterious when we realise that the two infinite series solutions are nothing more than the Taylor series for cos and sin respectively. That is

$$\cos \omega x = \sum_{n=0}^{\infty} \frac{(-1)^n(\omega x)^{2n}}{(2n)!},$$
$$\sin \omega x = \sum_{n=0}^{\infty} \frac{(-1)^n(\omega x)^{2n+1}}{(2n+1)!}.$$

By using power series we may solve a very large class of ordinary differential equations. Obviously the examples we have seen here are more easily solved by other methods. The power series method really comes into its own when we have to solve equations where the coefficients of the derivatives are functions of x.

3.2. **Ordinary Points.** To apply series methods to an ODE we need to distinguish between two different types of equations. Those with only ordinary points and those with regular singular points. First we discuss ordinary points.

Recall that a function f is analytic at a point x_0 if the Taylor series for f about x_0 is convergent in some interval I which contains x_0. More precisely

Definition 3.4. Let $f : I \to \mathbb{R}$ be infinitely differentiable at $x_0 \in I$. Let it have a Taylor series expansion about the point x_0 of the form

$$T_f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

If there is an interval $I_1 \subseteq I$ containing x_0 such that $f(x) = T_f(x)$ for all $x \in I_1$ then we say that f is *analytic at* x_0 or just *analytic*.

Examples of analytic functions which students are familiar with from high school are the trigonometric functions, the exponential function, all polynomials, the natural logarithm $\ln(1 + x)$ etcetara. Most of the functions encountered in first year mathematics courses are analytic on (at least) some finite interval.

Since we are trying to obtain Taylor expansions for solutions, the question of whether or not a function can be represented as a Taylor polynomial is crucial. It must be realised that not every function can be represented by a Taylor series. For example, the function $f(x) = |x|$ is not differentiable at $x = 0$, so it is not expressible as a Taylor series around zero.

A more interesting example is the function

$$f(x) = e^{-1/x^2}, x \neq 0, f(0) = 0.$$

Using the first principle definition of the derivative, we show that every derivative of f exsits at zero.

$$\begin{aligned} f'(0) &= \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{1}{x} e^{-1/x^2} \\ &= \lim_{u \to \infty} u e^{-u^2} = 0. \end{aligned}$$

Actually we can easily show that $f^{(n)}(0) = 0$ for each n. Hence the Taylor series of f about $x = 0$ is $T_f(x) = 0$. Clearly f is not equal to its Taylor series, except at zero.

Now we define ordinary points and singular points for an ODE.

Definition 3.5. Consider an nth order ODE of the form

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y(x) = 0 \tag{3.17}$$

The ODE (3.17) is said to have an *ordinary point* at x_0 if each of the functions $a_1..., a_n$ is analytic at x_0. A point which is not an ordinary point is said to be a *singular point*.

If we consider the ODE

$$y'' + \frac{2}{x} y' + y = 0, \tag{3.18}$$

then $x = 0$ is a singular point and all other points are ordinary points. Identifying ordinary and singular points is not usually a terribly difficult problem. Consider the equation

$$y'' + \frac{1}{\sin x} y' + \frac{1}{\cos x} y = 0. \tag{3.19}$$

This has singular points at $x = n\pi$ and $x = \frac{2n+1}{2}\pi$ for $n = 0, \pm 1, \pm 2, ...$ All other points are ordinary points.

The importance of this definition is the following theorem which we will not prove.

Theorem 3.6. *Consider the ODE*

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y(x) = 0. \tag{3.20}$$

Assume that each of the coefficient functions $a_1, ..., a_n$ is analytic at the point x_0 and is equal to its Taylor series expansion on the open interval I, where $x_0 \in I$. Then every solution of (3.20) is analytic at x_0 and equal to its Taylor series expansion on I.

A proof of this result in the $n = 2$ case may be found in Chapter 3 of Rabenstein, [4]. This Theorem means that if a differential equation has analytic coefficients then we can look for Taylor series solutions, at least on some suitable open interval.

Now let us look at an example where the ODE has a coefficient which is a power of x.

Example 3.4. Let us solve the ODE $y'' + xy' + y = 0$.
Solution First observe that the coefficient functions are all analytic for every value of x. That is, every value of x is an ordinary point for the ODE. We try a series solution as usual, setting $y = \sum_{n=0}^{\infty} a_n x^n$.

Differentiating twice and substituting into the equation gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n =$$
$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0. \tag{3.21}$$

To extract an expression for the coefficients a_n from this, notice that the middle series has first term $a_1 x$ and the first and third series have first terms $2a_2$ and a_0 respectively. So we take the constant terms from the first and third sum and write (3.21) as

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= 2a_2 + a_0 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=1}^{\infty} a_n x^n \\
&= 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + na_n + a_n\right] x^n \\
&= 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+1)a_n\right] x^n = 0. \qquad (3.22)
\end{aligned}$$

Since this can only be equal to zero if all the coefficients of the powers of x are zero, we conclude that $2a_2 + a_0 = 0$ and

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$$

or

$$(n+2)a_{n+2} + a_n = 0. \tag{3.23}$$

Starting with $a_2 = -1/2a_0$ we get for the even terms

$$a_4 = (-1)^2 \frac{1}{4} \times \frac{1}{2} a_0 = \frac{1}{2\times 2}\frac{1}{2\times 1} a_0 = \frac{1}{2^2 \times 1!} a_0,$$
$$a_6 = (-1)^3 \frac{1}{6} \times \frac{1}{4} \times \frac{1}{2} a_0 = -\frac{1}{2\times 3}\frac{1}{2\times 2}\frac{1}{2\times 1} a_0 = -\frac{1}{2^3 3!} a_0$$

It is easy to see that

$$a_8 = (-1)^4 \frac{1}{2^4 4!} a_0.$$

In general the even terms are given by

$$a_{2n} = (-1)^n \frac{1}{2^n n!} a_0.$$

For the odd coefficients we have starting with $n = 1$ in (3.23)

$$a_3 = -\frac{1}{3}a_1, a_5 = (-1)^2\frac{1}{5} \times \frac{1}{3}a_1, a_7 = (-1)^3\frac{1}{7} \times \frac{1}{5} \times \frac{1}{3}a_1,$$

etcetera.

To see the pattern here, rewrite the a_7 term as

$$a_7 = (-1)^3\frac{1}{7}\times\frac{1}{5}\times\frac{1}{3}a_1 = (-1)^3\frac{2\times 4\times 6}{1\times 2\times 3\times 4\times 5\times 6\times 7}a_1 = -(1)^3\frac{2^3 3!}{7!}a.$$

This generalises to give the odd coefficients as

$$a_{2n+1} = (-1)^n\frac{2^n n!}{(2n+1)!}a_1.$$

Rather than write the solution as a single series we split it into odd and even powers of x.

The solution of the ODE is given by the infinite series expansion

$$y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{2n} + a_1\sum_{n=0}^{\infty}\frac{(-1)^n 2^n n!}{(2n+1)!}x^{2n+1}. \qquad (3.24)$$

Expressing the second series as a known function is not so easy. However the first series is just the Taylor series for $e^{-x^2/2}$! The second series is actually a special function called the Error function.

The method has thus given us two linearly independent solutions of our ODE. The first is

$$y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{2n} = a_0 e^{-\frac{x^2}{2}}, \qquad (3.25)$$

and the second is

$$y(x) = a_1\sum_{n=0}^{\infty}\frac{(-1)^n 2^n n!}{(2n+1)!}x^{2n+1}. \qquad (3.26)$$

which is a more complicated function. In general, with a power series solution it is not easy to express it in terms of known functions and usually not necessary, since all the properties of the solution can in principle be obtained from the Taylor series.

Notice that these two infinite series converge for all values of x, which is a reflection of the fact that every point in $\mathbb{R}$ is a regular point for the ODE.

Example 3.5 (Airy's equation)**.** Find a power series solution of Airy's equation

$$y'' - xy = 0.$$

Solution Here every point is an ordinary point. So we again set $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. The differential equation then implies that

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0.$$

This is the same as

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= 2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= 2a_2 + \sum_{n=0}^{\infty}(n+3)(n+2)a_{n+3}x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\
2a_2 &+ \sum_{n=0}^{\infty}[(n+3)(n+2)a_{n+3} - a_n]x^{n+1} = 0.
\end{aligned}$$

As always, the coefficients of the powers of x must be equal to zero. So we immediately see that $a_2 = 0$. We also have

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)}.$$

Thus $a_5 = 0$, which implies $a_8 = 0$, $a_{11} = 0, a_{14} = 0$ etc. Consequently $a_{3n+2} = 0$.

Taking $n = 0$ gives $a_3 = \frac{1}{3\times 2}a_0$. We then get $a_6 = \frac{1}{6\times 5}\frac{1}{3\times 2}a_0$. Next we find

$$a_9 = \frac{1}{9\times 8}\frac{1}{6\times 5}\frac{1}{3\times 2}a_0 = \frac{7\times 4\times 1}{9\times 8\times ...\times 3\times 2\times 1}a_0 = \frac{7.4.1}{9!}a_0.$$

This gives $a_{12} = \frac{10.7.4.1.}{12!}a_0$. The general pattern is

$$a_{3n} = \frac{(3n-2)(3n-5)...4.1}{(3n)!}a_0,$$

for $n = 1, 2, 3, ..$

The remaining coefficients can be obtained the same way. We have the general expression

$$a_{3n+1} = \frac{(3n-1)(3n-4)...5.2}{(3n+1)!}a_1,$$

for $n = 1, 2, 3, ...$ The general solution of the differential equation can then be written

$$y = a_0(1+\sum_{n=1}^{\infty}\frac{1\cdot 4\cdots(3n-2)}{(3n)!}x^{3n}) + a_1(x+\sum_{n=1}^{\infty}\frac{2\cdot 5\cdots(3n-1)}{(3n+1)!}x^{3n+1}).$$

These two power series both converge for all values of x, which is again a reflection of the fact that every point in $\mathbb{R}$ is an ordinary point for the equation.

The functions defined by these power series are called Airy functions. In fact the Airy functions are defined as the sum and difference of these two series, weighted by certain constants.

Definition 3.7. The Airy functions *Ai* and *Bi* are the linearly independent solutions of the differential equation $y'' - xy = 0$. Define

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdots (3n-2)}{(3n)!} x^{3n}$$

and

$$g(x) = x + \sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdots (3n-1)}{(3n+1)!} x^{3n+1}.$$

Then the Airy functions of the first and second kind are defined as

$$Ai(x) = c_1 f(x) - c_2 g(x) \tag{3.27}$$

$$Bi(x) = \sqrt{3}\,[c_1 f(x) + c_2 g(x)] \tag{3.28}$$

The constants c_1 and c_2 are defined by $c_1 = 3^{-2/3}/\Gamma(2/3) \approx 0.355028$ and $c_2 = 3^{-1/3}/\Gamma(1/3) \approx 0.258819$

The Airy functions are very closely related to Bessel functions, which we will study later. They are named after the mathematician and astronomer George Airy (1801-1892) who discovered them while studying so called caustics in the theory of optics. A caustic is the envelope of light produced when light rays are reflected or refracted by a curved surface. Rainbows are the most familiar example.

Actually a rainbow is caused by the interaction of millions of caustics produced when light from the sun strikes drops of water in the atmosphere. Among Airy's other claims to fame is the establishment of the Greenwich meridian in 1851, in the city of Greenwich, England. This line is considered to have longitude zero and all others longitudes on earth are measured from this line.

The functions *Ai* and *Bi* behave very differently. The function *Ai* has the property that it is oscillatory for $x < 0$ and decays exponentially for $x > 0$. A plot of the function is given below.

The behaviour of *Ai* is interesting and in a sense predictable. The equation $y'' = ky$, where k is a constant, has exponential solutions for $k > 0$ - that is solutions $y = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$. Conversely it has oscillatory solutions for $k < 0$- that is solutions $y = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x)$. So we would expect the solutions of $y'' = xy$ to have similar behavior for $x > 0$ and $x < 0$. This is what we see. To the right of the origin, we have a function which decays exponentially and to the left, the function has decaying oscillations.

Conversely, the function *Bi* is unbounded. It grows exponentially for $x > 0$ and has small values for $x < 0$.

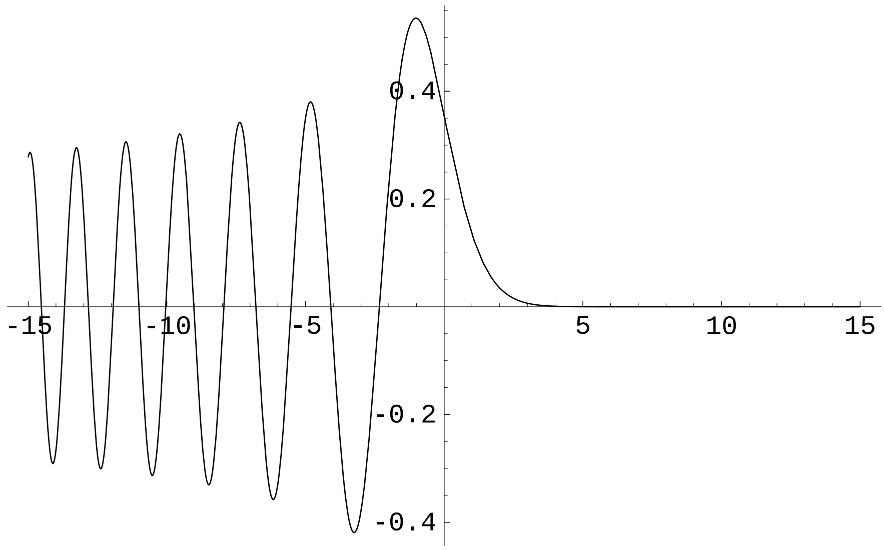

FIGURE 2. *Ai* The Airy function of the first kind

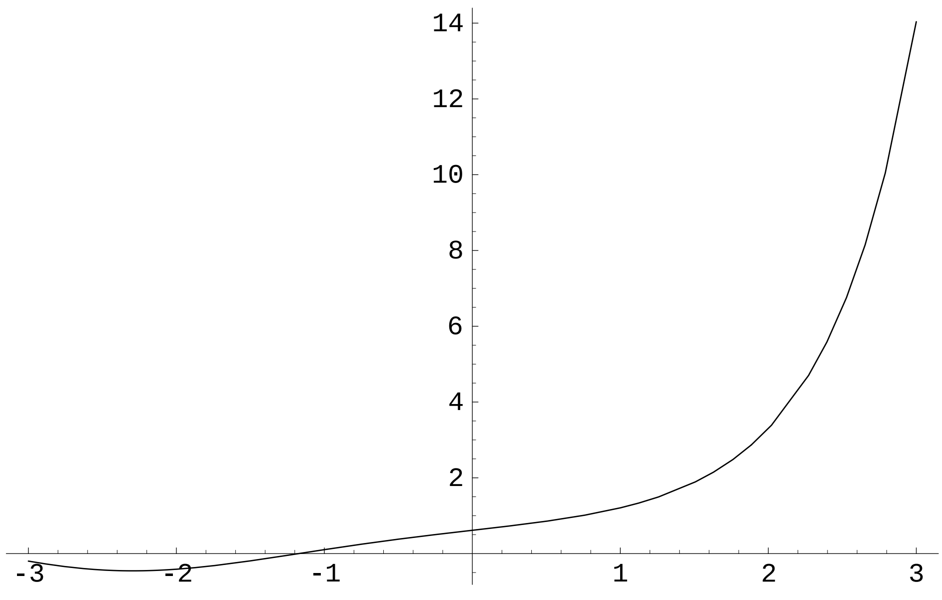

FIGURE 3. *Bi*, the Airy function of the second kind

One of the interesting features of special functions is the variety of ways in which they can be represented. Although we have a definition of *Ai* and *Bi* in terms of series solutions of the ODE in Example 4.5 it is often more convenient to use the following integral representation.

Proposition 3.8. *The Airy functions have the following representations as integrals*

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos(\frac{1}{3}t^3 + xt)dt \tag{3.29}$$

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left[\exp(-\frac{1}{3}t^3 + xt) + \sin(\frac{1}{3}t^3 + xt)\right] dt. \tag{3.30}$$

The Airy functions may also be written as combinations of Bessel functions, a result which will be presented after we discuss regular singular points.

Example 3.6. Solve the equation $(1 - x^2)y'' - 5xy' - 3y = 0$.
Solution. Here we observe that the equation has an ordinary point at zero, but two singular points at ± 1, since we can rewrite the ODE as

$$y'' - \frac{5x}{1-x^2}y' - \frac{3}{1-x^2}y = 0.$$

We therefore expect that the power series solution around $x = 0$ will only converge for $|x| < 1$. As before, we set $y = \sum_{n=0}^\infty a_n x^n$. Substituting into the differential equation produces

$$(1 - x^2)\sum_{n=2}^\infty n(n-1)a_n x^{n-2} - 5x\sum_{n=1}^\infty n a_n x^{n-1} - 3\sum_{n=0}^\infty a_n x^n =$$

$$\sum_{n=2}^\infty n(n-1)a_n x^{n-2} - \sum_{n=2}^\infty n(n-1)a_n x^n - \sum_{n=1}^\infty 5n a_n x^n - \sum_{n=0}^\infty 3a_n x^n = 0. \tag{3.31}$$

Remember that the guiding principle in this method is that the coefficient of every power of x must equal zero. So we collect the powers of x together by writing out the terms of the series, collecting like powers of x and setting their coefficients equal to zero, thus getting the appropriate expressions for the values of a_n. In this example the expression (3.31) can be rewritten as

$$2a_2 - 3a_0 + (6a_3 - 8a_1)x + \sum_{n=4}^\infty n(n-1)a_n x^{n-2}$$
$$- \sum_{n=2}^\infty [n(n-1) + 5n + 3]a_n x^n = 0. \tag{3.32}$$

The sum which starts at $n = 4$ can be shifted back to start at $n = 2$ by a change of index. Thus

$$\sum_{n=4}^\infty n(n-1)a_n x^{n-2} = \sum_{n=2}^\infty (n+2)(n+1)a_{n+2}x^n. \tag{3.33}$$

This means that equation (3.32) can be expressed as

$$2a_2 - 3a_0 + (6a_3 - 8a_1)x + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} - (n^2+4n+3)a_n]x^n$$
$$= 2a_2 - 3a_0 + (6a_3 - 8a_1)x + \sum_{n=2}^{\infty}(n+1)[(n+2)a_{n+2} - (n+3)a_n]x^n$$
$$= 0.$$

So the coefficients must satisfy $a_2 = \frac{3}{2}a_0$ and $a_3 = \frac{8}{6}a_1 = \frac{4}{3}a_1$. In general $(n+1)[(n+2)a_{n+2} - (n+3)a_n] = 0$. Cancelling the $n+1$ we have

$$a_{n+2} = \frac{n+3}{n+2}a_n.$$

Working out the values of the even coefficients recursively we have

$$a_4 = \frac{5}{4}a_2 = \frac{5}{4} \times \frac{3}{2}a_0, \; a_6 = \frac{7}{6} \times \frac{5}{4} \times \frac{3}{2}a_0, \; a_8 = \frac{9}{8} \times \frac{7}{6} \times \frac{5}{4} \times \frac{3}{2}a_0,$$

etcetera. Notice that $8.6.4.2 = 2^4(4.3.2.1)$. The general pattern is that for $n = 0, 1, 2, 3, ...$

$$a_{2n} = \frac{(2n+1)(2n-1)...3}{2^n n!}a_0.$$

For the odd coefficients, we get

$$a_5 = \frac{6}{5} \times \frac{4}{3}a_1, \; a_7 = \frac{8}{7} \times \frac{6}{5} \times \frac{4}{3}a_1, \; a_9 = \frac{10}{9} \times \frac{8}{7} \times \frac{6}{5} \times \frac{4}{3}a_1,$$

and so on. Now $10.8.6.4 = 2^4(5.4.3.2.1)$. Thus

$$a_{2n+1} = \frac{2^n(n+1)!}{(2n+1)(2n-1)...3.1}a_1.$$

As usual, it is convenient to separate the series into two series, one with even powers and one with odd powers. The series solution of the differential equation is therefore

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(2n+1)(2n-1)...3}{2^n n!}x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{2^n(n+1)!x^{2n+1}}{(2n+1)(2n-1)...3.1}.$$

It is not hard to check that these power series only converge for $|x| < 1$. This is exactly what we expected, since they are power series centered on zero and the differential equation has singular points at ± 1. We can think of the singular points as providing a 'barrier' to the series solution. Obtaining solutions for the differential equation which are valid outside the range $|x| < 1$ is possible, by looking for an expansion of the solution around a different point. For example, suppose that we wish to know the value of a function y satisfying the differential equation, and possibly some extra conditions, at $x = 2$. We could expand the solution about some point x_0, which is sufficiently close to 2 for the resulting power series to converge. This is an issue that we will not explore, however.

It is possible to prove some very useful facts about IVPs. The following is left as an exercise.

Theorem 3.9. *Let $p(x)$ and $q(x)$ be analytic in the interval $I = (-x_0, x_0)$, where $x_0 > 0$. Then the only solution on I to the IVP*

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

$y(0) = y'(0) = 0$ *is* $y(x) = 0$.

The power series method is a very useful tool. It does however have limitations. We have already seen that not every function can be expanded in a power series around a given point. We have seen two examples of this before. Let us consider an example of direct relevance to differential equations.

Example 3.7. The function $f(x) = \sqrt{x}$ cannot be represented by a power series around $x = 0$. To see why, suppose that it could be. Then for some coefficients $a_0, a_1, a_2...$, we would have

$$\sqrt{x} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots .$$

Setting $x = 0$ tells us that we must have $a_0 = 0$. Thus the series would have to be $\sqrt{x} = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$. Now suppose that $x \neq 0$. Then divide both sides by the square root of x. We would then have for all $x > 0$

$$1 = a_1\sqrt{x} + a_2x^{3/2} + a_3x^{5/2} + \cdots .$$

The left hand side is a constant, the right hand side is not a constant, so this is impossible. Thus $\sqrt{x}$ cannot be represented as a power series about $x = 0$. We could expand it as a Taylor series about another point, say $x = 1$, but not $x = 0$.

The relevance of this to the solution of differential equations is readily apparent if we consider the Euler equation

$$2x^2y'' + 3xy' - y = 0.$$

The reader may easily check that $y = \sqrt{x}$ is a solution of this equation. Clearly $x = 0$ is a singular point for the equation and we have a solution which cannot be represented as a power series around the singular point.

Fortunately there is a way around this if the singular point of the differential equation is a *regular singular point*

3.3. **Regular Singular Points and the Method of Frobenius.** We first start by defining a regular singular point. Then we will introduce the method of Frobenius which generalises the idea of a power series solution for a differential equation.

Definition 3.10. A differential equation

$$y^{(n)} + \frac{P_1(x)}{x-x_0}y^{(n-1)} + \cdots + \frac{P_{n-1}(x)}{(x-x_0)^{n-1}}y' + \frac{P_n(x)}{(x-x_0)^n}y = 0,$$

is said to have a *regular singular point at* x_0 if all of the coefficient functions P_i, $i = 1, ..., n$ are analytic at x_0.

As a simple example, the second order ordinary differential equation $2x^2y'' + 3xy' - y = 0$ has a regular singular point at $x = 0$. Another important example is provided by *Bessel's equation*,

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0,$$

where α is real. We will solve this important equation shortly. Each equation of this form has a regular singular point at $x = 0$.

The method of Frobenius depends on one important fact. Every differential equation with a regular singular point at x_0 has at least one solution of the form

$$y = (x-x_0)^s \sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

Here the number s need not be an integer.

The method thus extends the power series method in that we can find a solution as the product of a power series and some power of $(x - x_0)^s$. To illustrate we will present an example.

Example 3.8. Solve the differential equation

$$2x^2y'' + xy' - (x+1)y = 0,$$

by the method of Frobenius.

Solution. There is a regular singular point at $x = 0$. To see this observe that the equation can be rewritten as

$$y'' + \frac{x}{2x^2}y' - \frac{x+1}{2x^2}y = 0.$$

The functions x and $x + 1$ are analytic at $x = 0$, since they are polynomials and so 0 is a regular singular point.

Now we introduce the trial solution $y = \sum_{n=0}^{\infty} a_n x^{n+s}$. The first task is to identify the allowable values of s. We differentiate in the time honoured manner to get $y' = \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}$ and $y'' = \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2}$.

Notice that the differentiated series still starts at $n = 0$. The reason for this is the presence of the s in the exponent. If we have

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

then

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}. \tag{3.34}$$

So when we differentiate a regular power series, the constant term disappears, hence we can shift the starting value of n from 0 to 1 and we don't lose anything. Notice that if we left the index as starting at $n = 0$ in (3.34) then it would not make any difference because for $n = 0$ the term na_n is zero anyway.

However when we consider the method of Frobenius, things are a little different. Consider

$$y = \sum_{n=0}^{\infty} a_n x^{n+s} = a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + a_3 x^{s+3} + \cdots, \tag{3.35}$$

then

$$y' = s a_0 x^{s-1} + (s+1) a_1 x^s + (s+2) a_2 x^{s-1} + \cdots = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}. \tag{3.36}$$

There is no constant term in (3.35), so the index in the differentiated series still starts at $n = 0$.

Now we substitute the expressions for y' and y'' into the differential equation. The procedure is similar to what we do when finding a power series solution. We have in this case

$$\begin{aligned}
&2x^2 \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} + x \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} \\
&\quad - (x+1) \sum_{n=0}^{\infty} a_n x^{n+s} \\
&= \sum_{n=0}^{\infty} [2(n+s)(n+s-1) + (n+s) - 1] a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s+1} = 0.
\end{aligned} \tag{3.37}$$

Now the final series in the expression starts with the power x^{s+1}. The first series starts with a power x^s. So we take one term out of the first series so that the two series start with power x^{s+1}. We thus rewrite (3.37) as

$$\sum_{n=0}^{\infty}[2(n+s)(n+s-1)+(n+s)-1]a_nx^{n+s}-\sum_{n=0}^{\infty}a_nx^{n+s+1}$$
$$=(2s(s-1)+s-1)a_0x^s+\sum_{n=1}^{\infty}[2(n+s)(n+s-1)+(n+s)-1]a_nx^{n+s}$$
$$-\sum_{n=0}^{\infty}a_nx^{n+s+1}.$$

Notice that

$$\sum_{n=0}^{\infty}a_nx^{n+s+1}=\sum_{n=1}^{\infty}a_{n-1}x^{n+s}. \tag{3.38}$$

Hence we have

$$(2s(s-1)+s-1)a_0x^s+\sum_{n=1}^{\infty}[2(n+s)(n+s-1)+(n+s)-1]a_nx^{n+s}$$
$$-\sum_{n=1}^{\infty}a_{n-1}x^{n+s}=(2s^2-s-1)a_0x^s$$
$$+\sum_{n=1}^{\infty}[(2(n+s)(n+s-1)+(n+s)-1)a_n-a_{n-1}]x^{n+s}=0. \tag{3.39}$$

Observe that the terms inside the summation sign indicate that a_n is a multiple of a_{n-1}. Thus if $a_0=0$ then all the coefficients a_n will be zero! So we do not want $a_0=0$. This means that we must have $2s^2-s-1=0$, in order for the coefficient of x^s to equal zero. This gives us $s=1$ and $s=-1/2$ as the only possible choices.

First we will take $s=1$. Then from (3.39) we get

$$(2(n+s)(n+s-1)+(n+s)-1)a_n-a_{n-1}=(2n+3)na_n-a_{n-1}=0.$$

Hence

$$a_n=\frac{1}{n(2n+3)}a_{n-1}.$$

We may therefore generate the coefficients a_n recursively for $n\geq 1$. We have

$$a_1=\frac{1}{1.5}a_0,\ a_2=\frac{1}{2.7}\frac{1}{1.5}a_0,\ a_3=\frac{1}{3.9}\frac{1}{2.7}\frac{1}{1.5}a_0,\ a_4=\frac{1}{4.11}\frac{1}{3.9}\frac{1}{2.7}\frac{1}{1.5}a_0.$$

Thus $a_4=\frac{1}{11\times 9\times 7\times 5\times 4!}a_0$. In general for $n\geq 1$ we get

$$a_n=\frac{1}{n!(2n+3)(2n+1)...5}a_0=\frac{(2n+2)2n\cdots 4\cdot 3\cdot 2}{n!(2n+3)!}a_0$$
$$=3\frac{2^{n+1}(n+1)!}{n!(2n+3)!}a_0=3\frac{2^{n+1}(n+1)}{(2n+3)!}a_0.$$

So for $s = 1$ we have the solution

$$y_1 = a_0 \sum_{n=0}^{\infty} 3 \frac{2^{n+1}(n+1)}{(2n+3)!} x^{n+1}.$$

This series converges for all x. We can obviously absorb the 3 into the constant a_0.

If we take $s = -1/2$ we will get another solution. Using $s = -1/2$ we have

$$(2(n+s)(n+s-1) + (n+s) - 1)a_n - a_{n-1} = (2n^2 - 3n)a_n - a_{n-1} = 0.$$

Hence for $n \geq 1$ we have

$$a_n = \frac{1}{n(2n-3)} a_{n-1}.$$

Thus

$$a_1 = -a_0, a_2 = -\frac{1}{2.1} a_0, \; a_3 = -\frac{1}{3.2.1.1.3} a_0, \; a_4 = -\frac{1}{4.3.2.1.1.3.5} a_0$$

and so on. Continuing we see that for $n \geq 2$ the coefficients are given by

$$a_n = -\frac{1}{n!1.3...(2n-3)} a_0 = -\frac{2^{n-2}(n-2)!}{n!(2n-3)!} a_0 = -\frac{2^{n-2}}{n(n-1)(2n-3)!} a_0.$$

Hence for $s = -1/2$ we have the solution

$$y_{-1/2} = a_0 x^{-1/2} \left(1 - x - \sum_{n=2}^{\infty} \frac{2^{n-2}}{n(n-1)(2n-3)!} x^n \right).$$

The general solution to the differential equation is of course an arbitrary linear combination of y_1 and $y_{-1/2}$. Thus the general solution is

$$y = c_1 \sum_{n=0}^{\infty} \frac{2^{n+1}(n+1)}{(2n+3)!} x^{n+1} + c_2 x^{-1/2} \left(1 - x - \sum_{n=2}^{\infty} \frac{2^{n-2} x^n}{n(n-1)(2n-3)!} \right),$$

for arbitrary constants c_1 and c_2. (Note the factor of 3 in the solution y_1 has been absorbed into the constant c_1.) These solutions are valid for all $x > 0$. For $x < 0$ only y_1 is valid.

Notice that we have to write the second sum starting at $n = 2$ because our formula for a_n is only valid for $n \geq 2$. We separate out the a_0 and a_1 terms.

3.4. **The case of a single exponent.** The previous example illustrates the fact that for a second order ODE the exponent s is obtained in the method of Frobenius by finding the roots of a quadratic equation. It may be the case that the quadratic has only a single root s_1. In this case the method of Frobenius only produces one solution. However it is a relatively straightforward problem to produce a second solution from our known solution.

Let us assume that we have obtained a value $s = s_1$ and from this constructed a solution

$$y_1(x) = x^{s_1} \sum_{n=0}^{\infty} a_n(s_1) x^n. \tag{3.40}$$

Here we are emphasising the fact that coefficients a_n really depend on s. This was implicit in our previous example. Here we will make use of this fact. During the process of determining the possible values of the exponent s we obtain an expression $a_n(s)$ for the coefficients which depends on s. In the previous example we had

$$2(n+s)(n+s-1) + (n+s) - 1)a_n - a_{n-1}$$

giving

$$a_n(s) = \frac{a_{n-1}}{(n+s)^2 - 1}.$$

Taking the two values of s we found gave us two different sets of coefficients $a_n(s_1)$ and $a_n(s_2)$. In general, the expression for the coefficients $a_n(s)$ in the solution (3.40), when written in terms of s has the general form

$$a_n(s) = \frac{1}{(s+n-s_1)^2} \sum_{k=0}^{n} g_n(k,s) a_k, \quad n \geq 1, \tag{3.41}$$

for some function $g_n(k,s)$. This expression for $a_n(s)$ as a function of s can be used to derive another solution of the ODE by the following means.

We have the following theorem which we will not prove.

Theorem 3.11. *Let the ordinary differential equation*

$$y'' + p(x)y' + q(x)y = 0, \tag{3.42}$$

have a regular singular point at $x = 0$ and let $s = s_1$ be the only exponent obtained in the Method of Frobenius. Let the solution corresponding to s_1 be $y_1(x)$. Let $a_n(s)$ be given by (3.41). Then a second linearly independent solution of the ODE is

$$y_2(x) = y_1(x) \ln x + x^{s_1} \sum_{n=1}^{\infty} a_n'(s_1) x^n, \tag{3.43}$$

where $a_n'(s)$ is the derivative of $a_n(s)$ with respect to s.

Example 3.9. Find two linearly independent solutions of the ODE $x^2y'' - xy' + (1-x)y = 0$.

Solution. We look for a solution of the form $y = x^s \sum_{n=0}^{\infty} a_n x^n$. Substituting into the power series produces

$$x^2 \sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s-2} - x \sum_{n=0}^{\infty} (n+s)a_n x^{n+s-1} + \sum_{n=0}^{\infty} a_n x^{n+s} - x \sum_{n=0}^{\infty} a_n x^{n+s} = 0. \tag{3.44}$$

Taking the first term of the first three sums and noting that $(n+s)(n+s-1) - (n+s) - 1 = (n+s-1)^2$ we write this as

$$(s^2 - 2s + 1)a_0 x^s + \sum_{n=1}^{\infty} (n+s-1)^2 a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s+1} = 0.$$

The only root of $s^2 - 2s + 1 = 0$ is $s = 1$. Since $\sum_{n=0}^{\infty} a_n x^{n+s+1} = \sum_{n=1}^{\infty} a_{n-1} x^{n+s}$ we have, (making the dependence of a_n on s explicit again),

$$\sum_{n=1}^{\infty} [(n+s-1)^2 a_n(s) - a_{n-1}(s)] x^{n+s} = 0, \tag{3.45}$$

which tells us that $a_n(s) = \frac{1}{(n+s-1)^2} a_{n-1}(s)$. If we let $s = 1$ we get

$$a_n = \frac{1}{n^2} a_{n-1}.$$

Thus

$$a_1 = \frac{1}{1^2} a_0, \ a_2 = \frac{1}{2^2}\frac{1}{1^2} a_0, \ a_3 = \frac{1}{3^2}\frac{1}{2^2}\frac{1}{1^2} a_0, ...$$

Thus for $s = 1$ we have

$$a_n(s)|_{s=1} = \frac{1}{(n!)^2} a_0.$$

Our first solution is then

$$y_1(x) = x \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}.$$

Now we require a second solution and our theorem gives us this. We have $a_n(s) = \frac{1}{(n+s-1)^2} a_{n-1}$. So if we iterate this we get

$$a_n(s) = \frac{a_0}{s^2(s+1)^2 \cdots (s+n-1)^2}.$$

We require $a_n'(1)$. To obtain this we use logarithmic differentiation. That is, we take the log of both sides. We have

$$\begin{aligned} \ln a_n(s) &= \ln a_0 - \ln(s^2(s+1)^2 \cdots (s+n-1)^2) \\ &= \ln a_0 - 2\ln s - 2\ln(s+1) - \cdots - 2\ln(s+n-1). \end{aligned} \tag{3.46}$$

Differentiating both sides with respect to s we get

$$\frac{a_n'(s)}{a_n(s)} = -2(\frac{1}{s} + \frac{1}{s+1} + \cdots + \frac{1}{n+s-1}) \tag{3.47}$$

Thus, if we set $s = 1$ and recall that $a_n(1) = \frac{1}{(n!)^2}$, we obtain

$$a_n'(1) = -2a_n(1)\sum_{k=1}^{n}\frac{1}{k} = -\frac{2}{(n!)^2}H_n, \tag{3.48}$$

where H_n is the nth Harmonic number $H_n = \sum_{k=1}^{n}\frac{1}{k}$.

Thus from equation (3.43), we get the new solution

$$y_2(x) = y_1(x)\ln x - x\sum_{n=1}^{\infty}\frac{2H_n}{(n!)^2}x^n.$$

3.5. **When the exponents differ by an integer.** The third case we need to consider with the method of Frobenius is when the values of s we find differ by an integer. This is a slightly subtle problem. If there are two exponents s_1 and s_2 and $s_1 - s_2$ is an integer, there may be two linearly independent solutions which correspond to s_1 and s_2. However there are times when we do not get a second solution. The reason this can occur can be understood in the following manner.

Say that $s_2 = s_1 + N$, where N is an integer. Suppose also that we find y_1 corresponding to s_1 and $y_1 = x^{s_1}\sum_{n=0}^{\infty} a_n(s_1)x^n$.

Suppose that we try to find a second solution y_2. This would be of the form $y_2 = x^{s_2}\sum_{n=0}^{\infty} b_n(s_2)x^n$. but

$$x^{s_2}\sum_{n=0}^{\infty} b_n(s_2)x^n = x^{s_1}\sum_{n=0}^{\infty} b_n(s_2)x^{n+N}.$$

What can go wrong is that when we generate the terms $b_n(s_1)$ since s_1 and s_2 differ by an integer, the values of b_n may have essentially the same values as $a_n(s_1)$, just shifted by an integer N. Which would mean that

$$\sum_{n=0}^{\infty} b_n(s_2)x^{n+N} = C\sum_{n=0}^{\infty} a_n(s_1)x^n$$

for some constant C. In other words, the method of Frobenius may provide a solution y_1 for s_1, but for s_2 it just produces a scalar multiple of y_1. We will illustrate how this can happen in practice shortly.

So we want a second solution y_2 which is not a multiple of y_1. As in the case of equal exponents, there is a way of generating a second solution y_2 from a known solution y_1. We will again only present the result without proof.

Theorem 3.12. *Suppose that the ordinary differential equation*

$$y'' + p(x)y' + q(x)y = 0, \tag{3.49}$$

has a regular singular point at $x = 0$. Suppose that the exponents obtained by the method of Frobenius are s_1 and s_2 and $s_1 - s_2 = N$, where N is an integer. Further assume that there is a solution y_1 corresponding to the exponent s_1. Define $b_n(s) = (s - s_2)a_n(s)$. Then $\lim_{s\to s_2} b_N(s) = b_N \neq 0$ and a second linearly independent solution of (3.49) is

$$y_2(x) = \frac{b_N}{a_0} y_1(x) \ln x + x^{s_2} \sum_{n=0}^{\infty} b_n'(s_2) x^n. \tag{3.50}$$

We will now use the method of Frobenius to determine a solution of one of the most important differential equation, namely Bessel's equation. This will introduce the Bessel functions of the first kind, which are among the most important special functions in mathematics.

3.6. **Bessel's Equation.** The differential equation

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0 \tag{3.51}$$

is called Bessel's equation of order α. It has a regular singular point at $x = 0$. It was studied by Friedrich Bessel (1784-1846), a German astronomer and mathematician. We therefore would like a solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+s}$. Substitution of the trial solution into (3.51) gives

$$\sum_{n=0}^{\infty} [(n+s)^2 - \alpha^2] a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0. \tag{3.52}$$

Which is the same as

$$(s^2 - \alpha^2) a_0 x^s + ((s+1)^2 - \alpha^2) a_1 x^{s+1} + \sum_{n=2}^{\infty} [(n+s)^2 - \alpha^2] a_n x^{n+s}$$
$$+ \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0. \tag{3.53}$$

Since

$$\sum_{n=0}^{\infty} a_n x^{n+s+2} = \sum_{n=2}^{\infty} a_{n-2} x^{n+s}, \tag{3.54}$$

then (3.53) becomes

$$(s^2 - \alpha^2) a_0 x^s + ((s+1)^2 - \alpha^2) a_1 x^{s+1}$$
$$+ \sum_{n=2}^{\infty} [((n+s)^2 - \alpha^2)) a_n + a_{n-2})] x^{n+s} = 0. \tag{3.55}$$

Equating the coefficients of powers of x to zero gives $s^2 = \alpha^2$, so $s = \pm\alpha$. We also have to have $((s+1)^2 - \alpha^2) a_1 x^{s+1} = 0$, hence $a_1 = 0$. The remainder of the coefficients must satisfy $((n+s)^2 - \alpha^2) a_n + a_{n-2} = 0$.

If we take $s = \alpha$ then we have $(n^2 + 2n\alpha + \alpha^2 - \alpha^2)a_n + a_{n-2} = 0$. So for $n \geq 2$ we have

$$a_n = \frac{-1}{n^2 + 2n\alpha} a_{n-2}.$$

Generating the coefficients in the usual manner we see that all the odd coefficients are zero since $a_1 = 0$ and for the even coefficients we have

$$\begin{aligned}
a_2 &= -\frac{1}{2(2+2\alpha)} a_0 = -\frac{1}{2^2(1+\alpha)} a_0, \\
a_4 &= -\frac{1}{4(4+2\alpha)} a_2 = \frac{(-1)^2}{2^4 2!(1+\alpha)(2+\alpha)} a_0, \\
a_6 &= -\frac{1}{6(6+2\alpha)} a_4 = \frac{(-1)^3}{2^6 3!(1+\alpha)(2+\alpha)(3+\alpha)} a_0, \ldots
\end{aligned}$$

The general even coefficient is given by

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(1+\alpha)\cdots(n+\alpha)} a_0.$$

Thus the solution we obtain is

$$y_\alpha = a_0 x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!(1+\alpha)\cdots(n+\alpha)} x^{2n}.$$

For each a_0 we obviously have a solution. One very important choice for a_0 turns out to be

$$a_0 = \frac{1}{2^\alpha \Gamma(1+\alpha)}.$$

For this choice the solution y_α is called the Bessel function of the first kind. Recall that $\Gamma(x+1) = x\Gamma(x)$. Thus

$$(1+\alpha)(2+\alpha)\cdots(n+\alpha)\Gamma(1+\alpha) = \Gamma(n+\alpha+1).$$

Consequently the solution for this choice of α is as follows.

Definition 3.13. The Bessel function of the first kind of order α is defined by

$$J_\alpha(x) = x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+\alpha} n!\Gamma(n+\alpha+1)}. \tag{3.56}$$

The number α is called the *order* (or sometimes *index*) of the Bessel function.

We present a plot of the Bessel function J_1 below. From the graph we see that J_1 is an oscillatory function. Indeed all the Bessel functions of the first kind are oscillatory. This makes them important in the study of for example, wave motion.

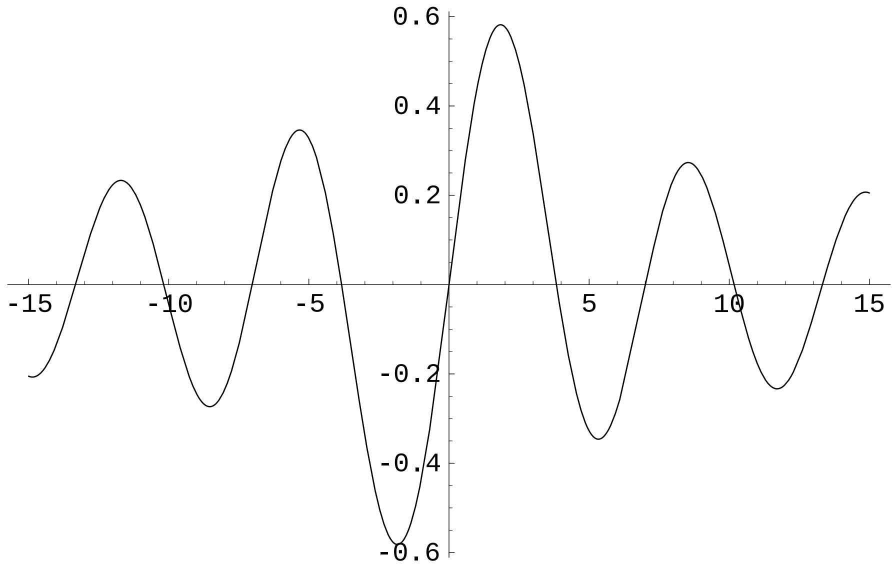

FIGURE 4. J_1, the Bessel function of the first kind of order 1.

By the same calculation as we have just presented it is possible to show that when α is not an integer taking $s = -\alpha$ leads to

$$J_{-\alpha}(x) = x^{-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! \Gamma(n - \alpha + 1)}.$$

In this case the functions $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly independent. Thus for α not an integer, the general solution of (3.51) is

$$y = c_1 J_\alpha(x) + c_2 J_{-\alpha}(x).$$

In the case when α is an integer $J_\alpha(x)$ and $J_{-\alpha}(x)$ are not linearly independent. In fact there is a very simple relationship between them. For n a positive integer we have $J_{-n}(x) = (-1)^n J_n(x)$. This is a consequence of the phenomenon discussed in the previous section. If α is an integer, then the exponents differ by an integer.

To see what goes wrong, we need to return to the calculation of the coefficients in the power series. We see that these are generated by the recurrence relation $((n + s)^2 - \alpha^2)a_n + a_{n-2} = 0$. For $s = -\alpha$ we have $[(n - \alpha)^2 - \alpha^2]a_n + a_{n-2} = 0$. But this is

$$(n^2 - 2n\alpha)a_n = -a_{n-2}.$$

Now if α is an integer, then 2α is also an integer. Suppose $2\alpha = N$. Then when we come to $n = N$ we have to satisfy the equation

$$(n^2 - 2n\alpha)a_n|_{n=N} = (N^2 - N^2)a_N = -a_{N-2}.$$

That is, $0 \cdot a_N = -a_{N-2}$. This requires $a_{N-2} = 0$. Which in turn forces $a_{N-4} = 0$. Indeed we require

$$a_0 = a_2 = \cdots = a_{N-2} = 0.$$

Thus the method essentially breaks down. We can obtain a solution by defining the Bessel function

$$J_{-N}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-N}}{2^{2n-N} n! \Gamma(n-N+1)}. \tag{3.57}$$

One can check that this is a solution. Notice that there is a term $1/\Gamma(n-N+1)$. The reciprocal of the Gamma function is an entire function, which means it is differentiable in the whole complex plain. Euler proved a remarkable formula, specifically

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-z/n} \right], \tag{3.58}$$

in which $\gamma = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{1}{k} - \ln m \right) = 0.57721....$ is Euler's constant. This is an infinite product involving the zeroes of the function. (Actually, any analytic function can be written as an infinite product involving its zeroes, but that is another subject).

It is obvious from this formula that $\dfrac{1}{\Gamma(-k)} = 0$ for k a positive integer. So the series (3.57) really starts at $n = N$. That is

$$J_{-N}(x) = \sum_{n=N}^{\infty} \frac{(-1)^n x^{2n-N}}{2^{2n-N} n! \Gamma(n-N+1)}. \tag{3.59}$$

Now we put $n = m + N$. This leads to

$$J_{-N}(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+N} x^{2m+N}}{2^{2m+N} n! \Gamma(m+N+1)} = (-1)^N J_N(x). \tag{3.60}$$

Thus J_N and J_{-N} are not actually linearly independent. So if α is an integer in order to obtain a second linearly independent solution of Bessel's equation we need to use Theorem 3.12.

Applying Theorem 3.12 to Bessel's equation when α is an integer leads to Bessel functions of the second kind. We will not go through the derivation of the second solution, we merely present the result.

Definition 3.14. Let n be an integer. Weber's Bessel function of the second kind $Y_n(x)$ is defined by

$$Y_n(x) = \frac{2}{\pi} \left[J_n(x)(\gamma + \ln \frac{x}{2}) - \frac{1}{2} \sum_{k=0}^{n-1} \frac{(n-k-1)!(x/2)^{2k-n}}{k!} \right.$$
$$\left. - \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k [H_k + H_{k+n}](x/2)^{2k+n}}{k!(k+n)!} \right]. \tag{3.61}$$

Here $H_n = \sum_{r=1}^{n} \frac{1}{r}$ and γ is Euler's constant. For α not an integer we can interpret Y_α by

$$Y_\alpha(x) = \frac{(\cos \pi\alpha) J_\alpha(x) - J_{-\alpha}(x)}{\sin \pi\alpha}. \tag{3.62}$$

It can also be shown that

$$Y_\alpha(x) = -\frac{2(x/2)^{-\alpha}}{\sqrt{\pi}\Gamma(1/2-\alpha)} \int_1^\infty \frac{\cos(xt)dt}{(t^2-1)^{\alpha+1/2}} dt. \tag{3.63}$$

An example of Y_n is plotted below. We now have the following result.

Proposition 3.15. *The general solution of Bessel's equation $x^2y'' + xy' + (x^2 - n^2)y = 0$ for integer n is $y = c_1 J_n(x) + c_2 Y_n(x)$.*

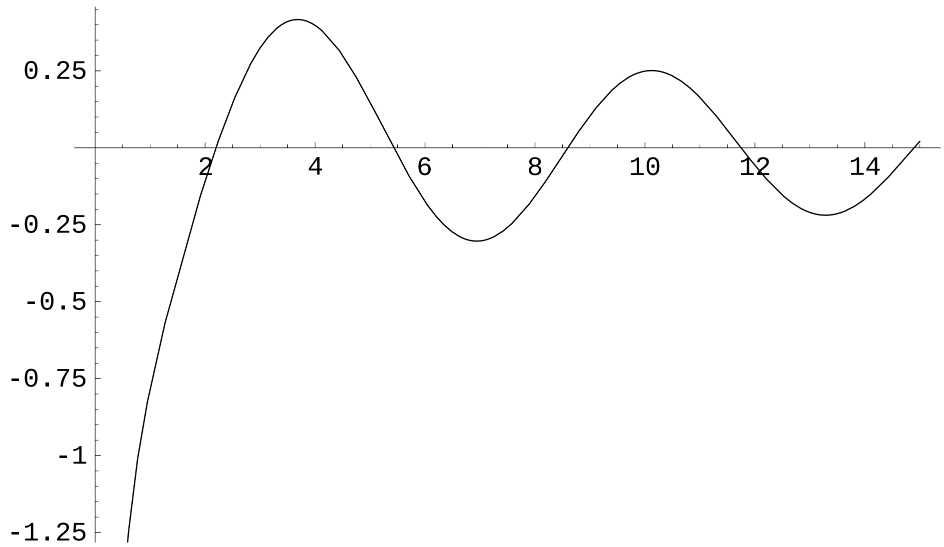

FIGURE 5. Y_1, the Bessel function of the second kind of order 1.

We will consider properties of Bessel functions in more detail in the next chapter, but before we do so, we will introduce the so called modified Bessel functions.

An equation closely related to the Bessel equation is the following

$$x^2y'' + xy' - (x^2 + \alpha^2)y = 0. \tag{3.64}$$

The solutions of this equation are called modified Bessel functions. They may be derived in the same way as J_α and Y_n. We state the results.

Proposition 3.16. *The modified Bessel equation*

$$x^2y'' + xy' - (x^2 + \alpha^2)y = 0. \tag{3.65}$$

has general solution $y = c_1 I_\alpha(x) + c_2 I_{-\alpha}(x)$ *when* α *is not an integer. Here*

$$I_\alpha(x) = x^\alpha \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k+\alpha} k! \Gamma(k+\alpha+1)} \tag{3.66}$$

is called a modified Bessel function of the first kind. If $\alpha = n$ *is an integer then the general solution is* $y = c_1 I_n(x) + c_2 K_n(x)$ *where the modified Bessel function of the second kind* K_n *is defined by*

$$K_n(x) = \frac{1}{2}(\frac{x}{2})^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(-\frac{x^2}{4})^k + (-1)^{n+1} I_n(x) \ln(\frac{x}{2})$$
$$+ (-1)^n \frac{1}{2}(\frac{x}{2})^n \sum_{k=0}^{\infty} \frac{\psi(k+1) + \psi(n+k+1)}{k!(n+k)!}(\frac{x^2}{4})^k. \tag{3.67}$$

Where $\psi(n) = -\gamma + \sum_{r=1}^{n-1} \frac{1}{r}$ *is the digamma function and* γ *is Euler's constant.*

3.6.1. *Differential equations solvable by Bessel functions.* A surprising number of differential equations can be solved by reducing them to Bessel equations. If we consider the general form of Bessel's equation $t^2 u'' + tu' + (t^2 - \alpha^2)u = 0$ and make the substitution $t = ax^r$ and $y = x^s u(ax^r)$ then Bessel's equation is transformed into the following ODE.

$$x^2 y''(x) + (1-2s)xy'(x) + [(s^2 - r^2\alpha^2) + a^2 r^2 x^{2r}]y(x) = 0. \tag{3.68}$$

The general solution of an ordinary differential equation of the form (3.68) is therefore $y = c_1 x^s J_\alpha(ax^r) + c_2 x^s Y_\alpha(ax^r)$, where we interpret Y_α to be $J_{-\alpha}$ in the case that α is not an integer.

This is valid for a purely imaginary. If a is purely imaginary we simply replace J_α with I_α and Y_α with K_α.

Example 3.10. Solve the differential equation

$$x^2 y'' + 7xy' + (4 + 36x^4)y = 0.$$

Solution Here $2r = 4$ so $r = 2$. Also $1 - 2s = 7$, so $s = -3$. Next we note that $s^2 - r^2\alpha^2 = 4$ which gives $\alpha = \pm\sqrt{5}/2$. Finally $a^2 r^2 = 36$, hence $a = 3$. Therefore the general solution is

$$y = x^{-3}[c_1 J_{\sqrt{5}/2}(3x^2) + c_2 J_{-\sqrt{5}/2}(3x^2)]. \tag{3.69}$$

Example 3.11. The differential equation $y'' + y = 0$ can be solved in terms of Bessel functions. We multiply by x^2 to obtain

$$x^2 y'' + x^2 y = 0.$$

This is of the general form we obtained with $1 - 2s = 0$, $s^2 - r^2\alpha^2 = 0, a^2 r^2 x^{2r} = x^2$. Hence $r = 1, s = 1/2$, and $\alpha = 1/2$. The general solution is thus

$$y = c_1\sqrt{x}J_{1/2}(x) + c_2\sqrt{x}J_{-1/2}(x). \tag{3.70}$$

We know that the general solution of this ODE is $y = A\sin x + B\cos x$. This suggests that $\sin x$ and $\cos x$ can be represented as Bessel functions. In the next chapter we will prove that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

Next we illustrate the connection between Bessel functions and Airy functions. The Airy equation is $y'' - xy = 0$. This is the same as $x^2y'' - x^3y = 0$. So $1 - 2s = 0$ giving $s = 1/2$. Also $2r = 3$ or $r = 3/2$. Next $a^2r^2 = -1$, or $a = \frac{2}{3}i$. Finally we obtain $\alpha = 1/3$. This tells us that the general solution of Airy's equation is

$$y = \sqrt{x}\left[c_1 I_{1/3}(\frac{2}{3}x^{3/2}) + c_2 I_{-1/3}(\frac{2}{3}x^{3/2})\right]. \tag{3.71}$$

Thus the Airy functions must be expressible in terms of Bessel functions. It is not hard to show by comparing (3.71) and the Airy functions for different choices of c_1, c_2 and x that the following result holds.

Proposition 3.17. *The Airy functions may be expressed in terms of Bessel functions by the following relations.*

$$Ai(x) = \frac{1}{3}\sqrt{x}[I_{-1/3}(\frac{2}{3}x^{3/2}) - I_{1/3}(\frac{2}{3}x^{3/2})] \tag{3.72}$$

$$Bi(x) = \sqrt{\frac{x}{3}}[I_{-1/3}(\frac{2}{3}x^{3/2}) + I_{1/3}(\frac{2}{3}x^{3/2})]. \tag{3.73}$$

3.6.2. *Bessel Functions and the Laplacian.* We take a look ahead at future material here. Given a twice differentiable function of two variables, $u(x, y)$ the Laplacian of u is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{3.74}$$

If we write the Laplacian in polar coordinates, $x = r\cos\theta, y = r\sin\theta$, then the Laplacian can be written

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2\theta}. \tag{3.75}$$

Suppose we want to solve the partial differential equation

$$\Delta u + k^2 u = 0.$$

This is an important equation in physics, since it arises in the study of gravity and electromagnetism. A technique which we will investigate in depth later in these notes is *separation of variables*. This is to look for a solution of the form $u(r, \theta) = R(r)\Theta(\theta)$.

Substituting this trial solution into the PDE gives

$$\Theta(R'' + \frac{1}{r}R' + k^2R) + \frac{R}{r^2}\Theta'' = 0. \tag{3.76}$$

Upon rearranging we have

$$\frac{r^2}{R}\left(R'' + \frac{1}{r}R' + k^2R\right) = -\frac{1}{\Theta}\Theta''. \tag{3.77}$$

The only way that a function of r can be equal to a function of θ for all r, θ is if both are constant. We thus have

$$\frac{r^2}{R}\left(R'' + \frac{1}{r}R' + k^2R\right) = \lambda$$

for some λ. This can be rearranged to give

$$r^2R'' + rR' + (k^2r^2 - \lambda)R = 0. \tag{3.78}$$

This is a form of Bessel's equation. Thus Bessel functions naturally arise when we solve problems involving the Laplacian in polar coordinates.

In the next chapter we will begin a deeper study of the properties of Bessel functions and discuss some of their properties.

4. Bessel functions

We have seen Bessel functions as solutions of the differential equation $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$. In this section we shall describe some of their properties. The Bessel functions $J_\alpha(x)$ are entire functions for $\alpha \geq 0$ and it is not hard to see from the power series

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^k x^{2k+\alpha}}{2^{2k+\alpha} k! \Gamma(k+\alpha+1)}$$

that

$$J_0(0) = 1, \quad J_\alpha(0) = 0, \quad \alpha > 0.$$

Many simple properties of Bessel functions can be derived from the power series. Here we present a few.

4.1. **Elementary properties of Bessel functions.**

Let us now derive some of the simplest but most useful properties of Bessel functions. We begin with some properties of the derivatives of Bessel functions. Observe that

$$\begin{aligned}
\frac{d}{dx}(x^\alpha J_\alpha(x)) &= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^k x^{2k+2\alpha}}{2^{2k+\alpha} k! \Gamma(k+\alpha+1)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^k (2k+2\alpha) x^{2k+2\alpha-1}}{2^{2k+\alpha} k! \Gamma(k+\alpha+1)} \\
&= x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^k (k+\alpha) x^{2k+\alpha-1}}{2^{2k+\alpha-1} k! \Gamma(k+\alpha+1)} \\
&= x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^k (k+\alpha) x^{2k+\alpha-1}}{2^{2k+\alpha-1} k! (k+\alpha) \Gamma(k+\alpha)} \\
&= x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^k x^{2k+\alpha-1}}{2^{2k+\alpha-1} k! \Gamma(k+\alpha)} = x^\alpha J_{\alpha-1}(x).
\end{aligned}$$

More or less the same argument shows that

$$\frac{d}{dx}(x^{-\alpha} J_\alpha(x)) = -x^{-\alpha} J_{\alpha+1}(x). \tag{4.1}$$

Now from the product rule we see that

$$\frac{d}{dx}(x^\alpha J_\alpha(x)) = \alpha x^{\alpha-1} J_\alpha(x) + x^\alpha J'_\alpha(x) = x^\alpha J_{\alpha-1}(x).$$

Rearranging this gives

$$J'_\alpha(x) = J_{\alpha-1}(x) - \frac{\alpha}{x} J_\alpha(x). \tag{4.2}$$

We also know that

$$\frac{d}{dx}(x^{-\alpha}J_\alpha(x)) = -\alpha x^{-\alpha-1}J_\alpha(x) + x^{-\alpha}J'_\alpha(x) = -x^{-\alpha}J_{\alpha+1}(x).$$

Upon rearranging this gives

$$J'_\alpha(x) = -J_{\alpha+1}(x) + \frac{\alpha}{x}J_\alpha(x) \tag{4.3}$$

Adding (4.2) and (4.3) gives us the relation

$$J'_\alpha(x) = \frac{1}{2}(J_{\alpha-1}(x) - J_{\alpha+1}(x)). \tag{4.4}$$

If on the other hand we subtract (4.2) from (4.3) we get

$$J_{\alpha+1}(x) = \frac{2\alpha}{x}J_\alpha(x) - J_{\alpha-1}(x). \tag{4.5}$$

We collect these together.

Proposition 4.1. *The Bessel functions of the first kind $J_\alpha(x)$ have the following properties.*

$$\frac{d}{dx}(x^{\alpha}J_\alpha(x)) = x^{\alpha}J_{\alpha-1}(x), \tag{4.6}$$

$$\frac{d}{dx}(x^{-\alpha}J_\alpha(x)) = -x^{-\alpha}J_{\alpha+1}(x), \tag{4.7}$$

$$J'_\alpha(x) = \frac{1}{2}(J_{\alpha-1}(x) - J_{\alpha+1}(x)), \tag{4.8}$$

$$J_{\alpha+1}(x) = \frac{2\alpha}{x}J_\alpha(x) - J_{\alpha-1}(x). \tag{4.9}$$

It is also quite easy to calculate simple integrals involving Bessel functions. One can show that

$$\int x^{\alpha}J_\alpha(x)dx = x^{\alpha+1}J_{\alpha+1}(x) + C$$

and

$$\int x^{1-\alpha}J_\alpha(x)dx = -x^{1-\alpha}J_{\alpha-1}(x) + C.$$

There are many integrals of Bessel functions which can be calculated exactly and we will present some examples later. However the reader can find hundreds of integrals of Bessel functions in [2].

Interestingly, the Bessel functions of the second kind $Y_\alpha(x)$ satisfy exactly the same relations, though this is a bit more complicated to prove. The modified Bessel functions of the first kind $I_\alpha(x)$ satisfy very similar relations. For example

$$I'_\alpha(x) = \frac{1}{2}(I_{\alpha-1}(x) + I_{\alpha+1}(x)).$$

These kinds of elementary observations are easy to show from the power series definition of $J_\alpha(x)$. However there are other ways of studying the Bessel functions, although all such methods ultimately descend from the power series definition. To conclude this section we derive the generating function for the Bessel functions $J_n(x)$ where n is an integer.

Theorem 4.2. *Let $J_n(x)$ be the Bessel function of the first kind of order n. Then we have the generating function*

$$e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(x) \qquad (t \neq 0). \tag{4.10}$$

Proof. The proof of the theorem is a not especially deep. We simply multiply the appropriate Taylor series together. We begin with the observation that

$$e^{ax} = 1 + ax + \frac{1}{2}a^2x^2 + \frac{1}{6!}a^3x^3 + \cdots .$$

Thus

$$e^{\frac{xt}{2}} = 1 + \frac{xt}{2} + \frac{1}{2!}\frac{x^2t^2}{2^2} + \frac{1}{3!}\frac{x^3t^3}{2^3} + \cdots$$

and

$$e^{-\frac{x}{2t}} = 1 - \frac{x}{2t} + \frac{1}{2!}\frac{x^2}{2^2t^2} - \frac{1}{3!}\frac{x^3}{2^3t^3} + \cdots$$

We now multiply these two series together and collect powers of t. The coefficient of t^0 is obtained by multiplying the terms with t^n in the numerator with the the correspond term with t^n in the denominator. The coefficient of t is obtained from multiplying the terms with t^n in the numerator with the terms that have t^{n-1} in the denominator and so on. So we have

$$\begin{aligned}
&e^{\frac{1}{2}x(t-\frac{1}{t})} = \\
&\left(1 + \frac{xt}{2} + \frac{1}{2!}\frac{x^2t^2}{2^2} + \frac{1}{3!}\frac{x^3t^3}{2^3} + \cdots\right)\left(1 - \frac{x}{2t} + \frac{1}{2!}\frac{x^2}{2^2t^2} - \frac{1}{3!}\frac{x^3}{2^3t^3} + \cdots\right) \\
&= 1 - \frac{x^2}{2^2} + \frac{x^4}{(2!)^2 2^4} - \frac{x^6}{(3!)^2 2^6} + \cdots + \frac{xt}{2}\left(1 - \frac{1}{2!}\frac{x^2}{2^2} + \frac{x^4}{2^4 2! 3!} - \cdots\right) \\
&- \frac{2}{xt}\left(1 - \frac{1}{2!}\frac{x^2}{2^2} + \frac{x^4}{2^4 2! 3!} - \cdots\right) + t^2(\frac{x}{2})^2\left(\frac{1}{2} - \frac{x^2}{2^2 4!} + \cdots\right)\cdots
\end{aligned} \tag{4.11}$$

Looking at the coefficients of t^n we see that they are nothing more than the Taylor expansion of $J_n(x)$. Hence

$$\begin{aligned}
e^{\frac{1}{2}x(t-\frac{1}{t})} &= J_0(x) + tJ_1(x) + \frac{1}{t}J_{-1}(x) + t^2J_2(x) + \frac{1}{t^2}J_2(x) + \cdots \\
&= \sum_{n=-\infty}^{\infty} t^n J_n(x).
\end{aligned}$$

This completes the proof. ∎

4.2. **Integral representations for Bessel functions.** As well as the generating function another important tool in the study of Bessel functions are integral representation of the Bessel functions. There are numerous representations of Bessel functions in terms of integrals. We will not attempt to give an exhaustive list. The book [1] has a large number of such representations. We will only present a few and indicate how they are derived.

Theorem 4.3. *The Bessel functions of the first kind of order n are given by*

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)d\theta \tag{4.12}$$

$$= \frac{i^{-n}}{\pi}\int_0^\pi e^{ix\cos\theta}\cos(n\theta)d\theta \tag{4.13}$$

Proof. We illustrate the proof of the first identity in the case $n = 0$. That is, we will show that

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)d\theta.$$

To prove this, we expand $\cos(x\sin\theta)$ in a Taylor series and integrate term by term. We have

$$\cos(x\sin\theta) = 1 - \frac{x^2\sin^2\theta}{2!} + \frac{x^4\sin^4\theta}{4!} + \frac{x^6\sin^2\theta}{6!} - \cdots . \tag{4.14}$$

So

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)d\theta \\
&= \frac{1}{\pi}\int_0^\pi \left(1 - \frac{x^2\sin^2\theta}{2!} + \frac{x^4\sin^4\theta}{4!} + \frac{x^6\sin^2\theta}{6!} - \cdots\right)d\theta \\
&= 1 - \frac{x^2}{2!}\frac{1}{\pi}\int_0^\pi \sin^2\theta d\theta + \frac{x^4}{4!}\frac{1}{\pi}\int_0^\pi \sin^4\theta d\theta + \cdots
\end{aligned} \tag{4.15}$$

We have to evaluate integrals of the form $\int_0^\pi \sin^{2k}\theta d\theta$, but these can all be done in closed form. The general result is formula 2.513.1 in [2].

$$\int \sin^{2n} x dx = \frac{1}{2^{2n}}\frac{(2n)!}{(n!)^2} + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\frac{2n!}{k!(2n-k)!}\frac{\sin(2n-2k)x}{2n-2k}$$

Since $\sin(2n-2k)\pi = 0$ for all integers n and k we have

$$\frac{1}{\pi}\int_0^\pi \sin^{2n} x dx = \frac{1}{2^{2n}}\frac{(2n)!}{(n!)^2}.$$

We therefore have

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)d\theta &= 1-\frac{x^2}{2!}\frac{2!}{2^2(1!)^2}+\frac{x^4}{4!}\frac{4!}{2^4(2!)^2}-\frac{x^6}{6!}\frac{6!}{2^6(3!)^2}-\cdots\\ &= 1-\frac{x^2}{2^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+\cdots\\ &= J_0(x). \qquad (4.16)\end{aligned}$$

We can proceed in the same manner for the general n case, but another method is to use the generating function. Observe that by Euler's formula

$$e^{ix\sin\phi}=\cos(x\sin\phi)+i\sin(x\sin\phi).$$

Now the generating function for J_n is

$$\begin{aligned}e^{\frac{1}{2}x(t-\frac{1}{t})} &= \sum_{n=-\infty}^{\infty}t^nJ_n(x)=J_0(x)+\sum_{n=-\infty}^{-1}t^nJ_n(x)+\sum_{n=1}^{\infty}t^nJ_n(x)\\ &= J_0(x)+\sum_{n=1}^{\infty}(t^nJ_n(x)+t^{-n}J_{-n}(x))\\ &= J_0(x)+\sum_{n=1}^{\infty}(t^n+(-1)^nt^{-n})J_n(x),\end{aligned}$$

since $J_{-n}(x)=(-1)^nJ_n(x)$. We put $t=e^{i\theta}$. Then

$$t-\frac{1}{t}=e^{i\theta}-e^{-i\theta}=2i\sin\theta,$$

which gives

$$e^{\frac{1}{2}x(t-\frac{1}{t})}=e^{ix\sin\theta}=J_0(x)+\sum_{n=1}^{\infty}(e^{in\theta}+(-1)^ne^{-in\theta})J_n(x).$$

For even n we have

$$e^{in\theta}+(-1)^ne^{-in\theta}=e^{in\theta}+e^{-in\theta}=2\cos(n\theta).$$

While for odd n

$$e^{in\theta}+(-1)^ne^{-in\theta}=e^{in\theta}-e^{-in\theta}=2i\sin(n\theta).$$

We therefore have

$$\begin{aligned}\cos(x\sin\theta)+i\sin(x\sin\theta) =& J_0(x)+\sum_{k=1}^{\infty}2\cos(2k\theta)J_{2k}(x)\\ &+2i\sum_{k=1}^{\infty}2\sin((2k-1)\theta)J_{2k-1}(x).\end{aligned}$$

Equating the real and imaginary parts gives

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{k=1}^{\infty}\cos(2k\theta)J_{2k}(x)$$
$$\sin(x\sin\theta) = 2\sum_{k=1}^{\infty}\sin((2k-1)\theta)J_{2k-1}(x).$$

To complete the proof we observe that

$$\cos(x\sin\theta - n\theta) = \cos(x\sin\theta)\cos(n\theta) + \sin(x\sin\theta)\sin(n\theta).$$

This however can be rewritten as

$$\begin{aligned}\cos(x\sin\theta - n\theta) = &\left(J_0(x) + \sum_{k=1}^{\infty}2\cos(2k\theta)J_{2k}(x)\right)\cos(n\theta)\\ &+ 2\sum_{k=1}^{\infty}2\sin((2k-1)\theta)J_{2k-1}(x)\sin(n\theta).\end{aligned}$$

Now it is simply a question of integrating.

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - n\theta)d\theta = &\frac{1}{\pi}J_0(x)\int_0^{\pi}\cos(n\theta)d\theta\\ &+ 2\sum_{k=1}^{\infty}J_{2k}(x)\frac{1}{\pi}\int_0^{\pi}\cos(2k\theta)\cos(n\theta)d\theta\\ &+ 2\sum_{k=1}^{\infty}J_{2k-1}(x)\frac{1}{\pi}\int_0^{\pi}\sin((2k-1)\theta)\sin(n\theta)d\theta.\end{aligned}$$

Again these trigonometric integrals are standard.

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi}\cos(2k\theta)\cos(n\theta)d\theta &= \frac{1}{\pi}\left[\frac{\sin[(2k-n)\theta]}{2(2k-n)} + \frac{\sin[(2k+n)\theta]}{2(2k+n)}\right]_0^{\pi}\\ &= \frac{1}{\pi}\begin{cases}0 & n\neq 2k\\ \frac{\pi}{2} & n = 2k\neq 0\\ \pi & n = 2k = 0\end{cases}.\end{aligned}$$

Similarly

$$\begin{aligned}&\frac{1}{\pi}\int_0^{\pi}\sin((2k-1)\theta)\sin(n\theta)d\theta\\ &= \frac{1}{\pi}\left[\frac{\sin[(2k-1-n)\theta]}{2(2k-1-n)} - \frac{\sin[(2k-1+n)\theta]}{2(2k-1+n)}\right]_0^{\pi}\\ &= \frac{1}{\pi}\begin{cases}0 & n\neq 2k-1\\ \frac{\pi}{2} & n = 2k-1\neq 0\end{cases}.\end{aligned}$$

So if $n = 2j$ for some integer $j \geq 1$ then

$$
\begin{aligned}
&2\sum_{k=1}^{\infty} J_{2k}(x)\frac{1}{\pi}\int_0^{\pi} \cos(2k\theta)\cos(n\theta)d\theta \\
&= 2\left(J_2(x)\frac{1}{\pi}\int_0^{\pi}\cos(2\theta)\cos(2j\theta)d\theta + J_4(x)\frac{1}{\pi}\int_0^{\pi}\cos(4\theta)\cos(2j\theta)d\theta + \right. \\
&\left. + \cdots + J_{2j}(x)\frac{1}{\pi}\int_0^{\pi}\cos(2j\theta)\cos(2j\theta)d\theta + \cdots\right) = 2\frac{\pi}{2}J_{2j}(x) = J_{2j}(x),
\end{aligned}
$$

since the only nonzero integral occurs when $2k = 2j$. Conversely all the integral of the form $\frac{1}{\pi}\int_0^{\pi}\sin((2k-1)\theta)\sin(n\theta)d\theta$ are zero when $n = 2j$. Thus for $n = 2j$

$$\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - 2j\theta)d\theta = J_{2j}(x) = J_n(x).$$

Now if $n = 2j+1$ all the cosine integrals are zero and only one of the sine integrals is nonzero, so we have that if $n = 2j-1$ then

$$\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - (2j-1)\theta)d\theta = J_{2j-1}(x) = J_n(x).$$

If $n = 0$ we obtain

$$\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)d\theta = J_0(x).$$

This completes the proof of the first integral identity. The second integral identity follows from the first by exploiting Euler's formula. □

There are also integral representations for Bessel functions of non integral order. We will only state without proof a few of these in the next theorem.

Theorem 4.4. *For $x > 0$ and $\Re(\alpha) > -\frac{1}{2}$, the following representations for the Bessel function of the first kind and the modified Bessel function of the first kind are valid.*

$$J_\alpha(x) = \frac{\left(\frac{x}{2}\right)^\alpha}{\sqrt{\pi}\Gamma(\frac{1}{2}+\alpha)}\int_0^{\pi}\cos(x\cos\theta)\sin^{2\alpha}\theta d\theta \tag{4.17}$$

$$I_\alpha(x) = \frac{\left(\frac{x}{2}\right)^\alpha}{\sqrt{\pi}\Gamma(\frac{1}{2}+\alpha)}\int_0^{\pi}e^{\pm x\cos\theta}\sin^{2\alpha}\theta d\theta. \tag{4.18}$$

For the Bessel function of the second kind we have for all complex numbers x with $|arg(x)| < \frac{1}{2}\pi$

$$
\begin{aligned}
Y_\alpha(x) &= \frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta - \alpha\theta)d\theta \\
&- \frac{1}{\pi}\int_0^{\infty}(e^{\alpha t} + e^{-\alpha t}\cos\alpha\pi)e^{-x\sinh t}dt.
\end{aligned} \tag{4.19}
$$

The book [1] has many more such representations. Integral representations of Bessel functions, or indeed any special function can be a very powerful tool in the analysis of many problems. It is often easier to deduce a property of a function from its integral representation than it is to deduce it from the power series expansion.

4.3. **Connections between Bessel functions and other functions.** The Bessel functions are solutions of a certain kind of ordinary differential equation. We have seen that the Bessel equation can be transformed into a rather more general form by a simple change of variables. This leads to the observation that certain kinds of functions can be expressed in terms of Bessel functions and certain Bessel functions can be expressed in terms of other functions.

We will begin with the trigonometric functions. It turns out that the Bessel functions of order $\frac{(2n+1)}{2}$ for all integer n can be expressed in terms of the standard trigonometric functions.

Proposition 4.5. *The Bessel functions $J_{1/2}$ and $J_{-1/2}$ satisfy*

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \tag{4.20}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x. \tag{4.21}$$

Similarly we have

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x, \tag{4.22}$$

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x. \tag{4.23}$$

Proof. We will only do the case of $J_{1/2}$. The other cases are similar. We have the Taylor series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Now we calculate the Taylor series for $J_{1/2}$.

$$\begin{aligned} J_{1/2}(x) &= (\frac{x}{2})^{\frac{1}{2}}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{2^{2k}k!\Gamma(k+1+\frac{1}{2})} \\ &= (\frac{x}{2})^{\frac{1}{2}}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{2^{2k}k!(k+\frac{1}{2})\Gamma(k+\frac{1}{2})}, \end{aligned}$$

since $\Gamma(a+1) = a\Gamma(a)$. Applying this property of the Gamma function k times we have

$$\begin{aligned}\Gamma(k+1+\frac{1}{2}) &= (k+\frac{1}{2})(k-\frac{1}{2})(k-\frac{3}{2})\cdots\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})\\ &= (k+\frac{1}{2})(k-\frac{1}{2})(k-\frac{3}{2})(k-\frac{5}{2})\cdots\frac{3}{2}\frac{1}{2}\sqrt{\pi}.\end{aligned}$$

So expanding the Taylor series we have

$$\begin{aligned}J_{1/2}(x) &= (\frac{x}{2})^{\frac{1}{2}}\left(\frac{2}{\sqrt{\pi}} - \frac{x^2}{2^2\frac{3}{2}\frac{1}{2}\sqrt{\pi}} + \frac{x^4}{2^4 2!\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}} - \cdots\right)\\ &= (\frac{x}{2})^{\frac{1}{2}}\left(\frac{2}{\sqrt{\pi}} - \frac{2x^2}{3.2.1\sqrt{\pi}} + \frac{2x^4}{5.4.3.2.1} - \cdots\right)\\ &= \sqrt{\frac{2}{\pi}}\sqrt{x}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)\\ &= \sqrt{\frac{2}{\pi}}\left(\sqrt{x} - \frac{x^{5/2}}{3!} + \frac{x^{7/2}}{5!} - \cdots\right)\\ &= \sqrt{\frac{2}{\pi x}}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\\ &= \sqrt{\frac{2}{\pi x}}\sin x.\end{aligned}$$

□

For an arbitrary integer n we have

Proposition 4.6. *For $n = 0, \pm 1, \pm 2, ...$ we have*

$$\sqrt{\frac{\pi x}{2}}J_{n+\frac{1}{2}}(x) = f_n(x)\sin x + (-1)^{n+1}f_{-n-1}(x)\cos x, \tag{4.24}$$

where $f_0(x) = 1/x,\ f_1(x) = x^{-2}$ and

$$f_{n-1}(x) - f_{n+1}(x) = (2n+1)\frac{1}{x}f_n(x).$$

Equivalently

$$\sqrt{\frac{\pi x}{2}}J_{n+\frac{1}{2}}(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sin x}{x}. \tag{4.25}$$

For the Bessel functions of the third kind we have

$$\sqrt{\frac{\pi x}{2}}Y_{n+\frac{1}{2}}(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\frac{\cos x}{x}. \tag{4.26}$$

We also have

$$\sqrt{\frac{\pi x}{2}}I_{n+\frac{1}{2}}(x) = g_n(x)\sinh x + g_{-n-1}(x)\cosh x, \tag{4.27}$$

where $g_0(x) = 1/x,\ g_1(x) = -x^{-2}$ *and*

$$g_{n-1}(x) - g_{n+1}(x) = (2n+1)\frac{1}{x}g_n(x).$$

This is equivalent to

$$\sqrt{\frac{\pi x}{2}}I_{n+\frac{1}{2}}(x) = x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sinh x}{x}. \tag{4.28}$$

The proof of this theorem can be obtained by using the integral representations for the Bessel functions. We omit it and simply present an example. For $n = 1$ we obtain from the theorem.

$$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{\sinh x}{x^2} + \frac{\cosh x}{x}\right).$$

The reader can easily generate the corresponding expressions for $J_{3/2}$ and $Y_{3/2}$.

4.4. **Integrals involving Bessel functions.** We have seen some simple integrals of Bessel functions. Now we will evaluate some more complicated examples. There are many areas where it is necessary to evaluate an integral involving a Bessel function. For example, in bond pricing, one often needs to compute an integral involving modified Bessel functions of the first kind multiplied by some other function, in order to obtain the price of a bond or an option on a bond. Many of these integrals can be done in Mathematica, but it is important to be aware of how some of these integrals are actually done.

We first consider the following problem. The Laplace transform (denoted by either $\overline{f}$ or F) of a suitable function f is defined by

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

It will be studied in more detail later. Here we calculate an important example.

Example 4.1. Calculate the Laplace transform of J_0. That is, evaluate the integral

$$\overline{J_0}(s) = \int_0^\infty e^{-st}J_0(x)dx. \tag{4.29}$$

Solution. We know that

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)d\theta.$$

So the Laplace transform is

$$\begin{aligned}\overline{J_0}(s) &= \frac{1}{\pi}\int_0^\infty\int_0^\pi e^{-sx}\cos(x\sin\theta)d\theta dx\\ &= \frac{1}{\pi}\int_0^\pi\int_0^\infty e^{-sx}\cos(x\sin\theta)dxd\theta \qquad (4.30)\end{aligned}$$

where we have reversed the order of integration. The inner integral can now be done using integration by parts. We have

$$\int_0^\infty e^{-sx}\cos(x\sin\theta)dx = \frac{s}{s^2+\sin^2\theta}. \tag{4.31}$$

This gives

$$\overline{J_0}(s) = \frac{1}{\pi}\int_0^\pi \frac{s}{s^2+\sin^2\theta}d\theta. \tag{4.32}$$

This last integral is one that is perhaps not familiar, however it can be done in a number of ways. One way is to use residues but it is not necessary. An antiderivative can be found. The substitution $z = \cot\theta$ will reduce the integral to a readily computable form. For the lazy mathematician however looking at the table of integrals in [2], specifically formula 2.562.1 gives us the result we require.

$$\frac{1}{\pi}\int \frac{s}{s^2+\sin^2\theta}d\theta = \frac{1}{\sqrt{s^2+1}}\tan^{-1}\left(\sqrt{\frac{1+s^2}{s^2}}\tan\theta\right). \tag{4.33}$$

There is a slight subtlety here, however. The range of integration we want is $[0,\pi]$, but the function $\tan\theta$ is singular at $\theta = \pi/2$ and zero at both endpoints. However we observe that

$$\frac{1}{\pi}\int_0^\pi \frac{s}{s^2+\sin^2\theta}d\theta = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{s}{s^2+\sin^2\theta}d\theta + \frac{1}{\pi}\int_{\frac{\pi}{2}}^\pi \frac{s}{s^2+\sin^2\theta}d\theta$$

Since $\sin(\pi - t) = \sin t$, then if we put $\theta = \pi - t$ we get

$$\begin{aligned}
\frac{1}{\pi}\int_0^\pi \frac{s}{s^2+\sin^2\theta}d\theta &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{s}{s^2+\sin^2\theta}d\theta - \frac{1}{\pi}\int_{\frac{\pi}{2}}^0 \frac{s}{s^2+\sin^2 t}dt \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{s}{s^2+\sin^2\theta}d\theta \\
&= \frac{2}{\pi}\lim_{R\to\frac{\pi}{2}}\left[\frac{1}{\sqrt{s^2+1}}\tan^{-1}\left(\sqrt{\frac{1+s^2}{s^2}}\tan\theta\right)\right]_0^R \\
&= \frac{1}{\sqrt{1+s^2}} \qquad (4.34)
\end{aligned}$$

since $\tan R \to \infty$ as $R \to \frac{\pi}{2}$ and $\tan^{-1}(z) \to \frac{\pi}{2}$ as $z \to \infty$.

As an extension of this we can easily do the following.

Example 4.2. Show that

$$\int_0^\infty J_0(x)dx = 1$$

for all positive integers n.

Solution. We know that

$$\int_0^\infty J_0(x)e^{-sx}dx = \frac{1}{\sqrt{1+s^2}} \tag{4.35}$$

We simply take $s = 0$. We can generalise this by the change of variables $z = bx$ to show that for $b > 0$

$$\int_0^\infty J_0(bx)dx = \frac{1}{b}.$$

By another change of variables we can easily show that

$$\int_0^\infty J_0(bx)e^{-sx}dx = \frac{1}{\sqrt{s^2+b^2}}.$$

A rather involved calculation with the power series expansion for $J_n(x)$ shows that

$$\int_0^\infty e^{-sx}J_n(bx)x^n dx = \frac{2^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}\frac{b^n}{(a^2+b^2)^{n+\frac{1}{2}}}. \tag{4.36}$$

Finally, setting $s = ia$, $n = 0$ and taking the real and imaginary parts we can establish the following.

$$\int_0^\infty \sin ax J_0(bx)dx = \begin{cases} 0 & b > a \\ \frac{1}{\sqrt{a^2-b^2}} & b < a, \end{cases} \tag{4.37}$$

$$\int_0^\infty \cos ax J_0(bx)dx = \begin{cases} 0 & b < a \\ \frac{1}{\sqrt{b^2-a^2}} & b > a. \end{cases} \tag{4.38}$$

Actually, many integrals of special functions are calculated by manipulating the relevant series expansion. We will do an example of such a calculation.

Example 4.3. Evaluate *Weber's integral* to prove that

$$\int_0^\infty e^{-a^2x^2}J_\alpha(bx)x^{\alpha+1}dx = \frac{b^\alpha}{(2a^2)^{\alpha+1}}e^{-b^2/4a^2} \tag{4.39}$$

Solution. To do this we use the series expansion for $J_\alpha(x)$ and reverse the order of the sum and the integral. So we have

$$\begin{aligned}\int_0^\infty e^{-a^2x^2}J_\alpha(bx)x^{\alpha+1}dx &= \int_0^\infty e^{-a^2x^2}x^{\alpha+1}\sum_{k=0}^\infty \frac{(-1)^k(bx/2)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)}dx \\ &= \sum_{k=0}^\infty \frac{(-1)^k(b/2)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)}\int_0^\infty e^{-a^2x^2}x^{2\alpha+2k+1}dx.\end{aligned} \tag{4.40}$$

We now use the substitution $t = a^2x^2$ in the integral. So $xdx = dt/2a^2$. This converts the integral into a Gamma function. We have

$$\int_0^\infty e^{-a^2x^2}x^{2\alpha+2k}xdx = \frac{1}{2a^{2(k+\alpha+1)}}\int_0^\infty e^{-t}t^{\alpha+k}dt = \frac{\Gamma(\alpha+k+1)}{2a^{2(k+\alpha+1)}}.$$

This gives us

$$\int_0^\infty e^{-a^2x^2} J_\alpha(bx)x^{\alpha+1}dx = \sum_{k=0}^\infty \frac{(-1)^k(b/2)^{2k+\alpha}}{k!}\frac{1}{2a^{2(k+\alpha+1)}}$$
$$= \frac{b^\alpha}{(2a^2)^{(\alpha+1)}}\sum_{k=0}^\infty \frac{(-1)^k(b^2/4a^2)^k}{k!}$$
$$= \frac{b^\alpha}{(2a^2)^{(\alpha+1)}}e^{-b^2/4a^2}, \tag{4.41}$$

since the last sum is just the Taylor series for e^{-u} with $u = b^2/4a^2$.

Integrals like this come up a great deal in the theory of Bessel processes and in particular, so called square root models for interest rates. However it is often necessary to evaluate the integrals with the $J_\alpha(x)$ replaced by $I_\alpha(x)$. We notice that the series expansion for I_α is the same as that for J_α with the difference that there is no $(-1)^k$ term. Thus we can repeat the calculation for the modified Bessel function and obtain

$$\int_0^\infty e^{-a^2x^2} I_\alpha(bx)x^{\alpha+1}dx = \frac{b^\alpha}{(2a^2)^{(\alpha+1)}}\sum_{k=0}^\infty \frac{(b^2/4a^2)^k}{k!}$$
$$= \frac{b^\alpha}{(2a^2)^{(\alpha+1)}}e^{b^2/4a^2}. \tag{4.42}$$

We can generalise these results by further manipulation of the series. For example we can prove the following.

$$\int_0^\infty e^{-a^2x^2} J_\alpha(bx)x^{\alpha+3}dx = \frac{b^\alpha}{2^{\alpha+1}a^{\alpha+2}}\left(\alpha+1-\frac{b^2}{4a}\right)e^{-b^2/4a} \tag{4.43}$$

for $a > 0$.

The Bessel functions $J_\alpha(x)$ have infinitely many zeroes on the positive real axis. Let us call these zeroes $\{\xi_i\}_{i=1}^\infty$. It is possible to compute the zeroes numerically and these have been tabulated, in for example [1]. Using a package such as Mathematica we can generate as many zeroes as we desire. Our concern here however is with the following result.

Proposition 4.7. *The Bessel functions $J_n(\xi_i x)$, satisfy*

$$\int_0^a xJ_n(\xi_i x)J_n(\xi_j x)dx = \frac{a^2}{2}[J_{n+1}(\xi_i a)]^2\delta_{ij}, \tag{4.44}$$

where ξ_i and ξ_j are distinct roots of the equation $J_n(\xi a) = 0$. Here δ_{ij} is the Kroneckor delta. That is $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

This result is extremely important. It establishes the fact that the Bessel functions are *orthogonal* in some sense. This means that one can expand functions in series of Bessel functions. The expansion of functions in series of so called *orthogonal functions* is a major area in its own right.

4.5. **Bessel functions of the third kind.** To conclude our study of Bessel functions we will briefly mention the Bessel functions of the third kind. These are actually not new Bessel functions at all. They are in fact just Bessel functions of the first and second kind taken in complex linear combination. More precisely we have

Definition 4.8. The Bessel functions of the third kind, also known as the Hankel functions of the first and second kind are

$$H^1_\alpha(x) = J_\alpha(x) + iY_\alpha(x) \tag{4.45}$$

$$H^2_\alpha(x) = J_\alpha(x) - iY_\alpha(x) \tag{4.46}$$

Since these functions are just linear combinations of the first and second kind Bessel functions, one can deduce their properties from studying J_α and Y_α. Consequently we will not say any more about them.

5. Laplace Transforms

Suppose that f is a function defined on $[0, \infty)$ such that the integral $\int_0^\infty f(t)e^{-st}dt$ converges for $s \in \Omega \subset \mathbb{C}$, where

$$\Omega = \{s \in \mathbb{C} | 0 \le s_1 \le Re(s) \le s_2 \le \infty\}.$$

Here $Re(s)$ is the real part of s. Then the Laplace transform of f is defined by

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}dt, \ s \in \Omega. \tag{5.1}$$

A sufficient condition for the Laplace transform of f to exist is that f is continuous and there exists constants $K > 0$ and a such that $|f(t)| \le Ke^{at}$.

Often we denote the Laplace transform by using the corresponding capital letter. Thus

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}dt. \tag{5.2}$$

The Laplace transform is closely related to the so called *Fourier transform*. The Fourier transform of a function f is defined by the integral

$$\mathcal{F}(f)(y) = \int_{-\infty}^\infty f(t)e^{-ity}dt. \tag{5.3}$$

The Fourier transform is one of the most important tools in analysis. In a loose sense it can be thought of as "arranging" the information in a function in terms of frequencies instead of evolution over time. It plays a major role in probability theory, differential equations, signal and image processing and many other areas. For example, in probability theory, the Fourier transform is also known as the characteristic function for a continuous random variable.

If we assume that $f(t) = 0$ for all $t < 0$, then setting $y = -si$ gives

$$\mathcal{F}(f)(s) = \int_0^\infty f(t)e^{-st}dt. \tag{5.4}$$

So $\mathcal{L}(f)$ is a special case of the Fourier transform, in which the transform variable s is considered to be a complex variable. Many properties of the Laplace transform can be deduced from corresponding properties of the Fourier transform. For example, there is a result known as the Plancherel-Parseval Theorem which tells us that if f is a square integrable function then

$$\int_{-\infty}^\infty |\widehat{f}(t)|^2 dt = 2\pi \int_{-\infty}^\infty |f(t)|^2 dt.$$

There is also an inversion theorem for the Fourier transform, which allows us to recover the original function from the transform. Specifically, if $\widehat{f}$ is integrable, then we recover f by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(y)e^{iyt}dy. \tag{5.5}$$

The corresponding results for Laplace transform can be obtained from those for the Fourier transform. See any standard text on Fourier analysis for a discussion of this.

However it would be wrong just to think of the Laplace transform as simply a different version of the Fourier transform. It has its own unique and interesting features, which make it well worth studying in its own right. Let us begin by presenting some elementary examples.

Example 5.1. Find the Laplace transform of $f(t) = 1$.
Solution By definition of the Laplace transform we have

$$\mathcal{L}(f)(s) = \int_0^{\infty} e^{-st}dt = \left[-\frac{1}{s}e^{-st}\right]_0^{\infty} = \frac{1}{s}. \tag{5.6}$$

Example 5.2. Calculate the Laplace transform of $f(t) = t^n$.
Solution. We again integrate by parts to obtain

$$\begin{aligned}\mathcal{L}(f)(s) = \int_0^{\infty} t^n e^{-st}dt \qquad &= \left[-\frac{1}{s}t^n e^{-st}\right]_0^{\infty} + \frac{n}{s}\int_0^{\infty} t^{n-1}e^{-st}dt \\ = \frac{n}{s}\mathcal{L}(t^{n-1})(s). & \end{aligned} \tag{5.7}$$

From this it follows that

$$\mathcal{L}(t^n)(s) = \frac{n}{s}\mathcal{L}(t^{n-1})(s) = \frac{n(n-1)}{s^2}\mathcal{L}(t^{n-2}) = \cdots = \frac{n!}{s^{n+1}}.$$

Example 5.3. Now let us calculate the Laplace transform of $f(t) = t^a, a > -1$. The restriction on a is to ensure that the integral converges. We have via the change of variables $st = u$

$$\begin{aligned}\int_0^{\infty} t^a e^{-st}dt &= \frac{1}{s^{a+1}}\int_0^{\infty} u^a e^{-u}du \\ &= \frac{\Gamma(a+1)}{s^{a+1}}.\end{aligned}$$

So $\mathcal{L}(f)(s) = \dfrac{\Gamma(a+1)}{s^{a+1}}$.

5.1. **Elementary Properties of the Laplace Transform.**

The most basic fact about the Laplace transform is that it is linear.

Proposition 5.1. *The Laplace transform is a linear operator. That is, if $\mathcal{L}(f)(s) = F(s)$ and $\mathcal{L}(g)(s) = G(s)$, for all $s \in \Omega$, then for all constants a, b, we have*

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g).$$

Proof. The proof follows from the fact that integration is linear. □

A useful tool for calculating Laplace transforms is to differentiate under the integral sign. We are able to do this because of a very deep property of the Laplace transform.

Theorem 5.2. *Let f be a piecewise continuous function, such that $F(s) = \int_0^\infty f(t)e^{-st}dt$, for $s \in \Omega \subset \mathbb{C}$. Then F is an analytic function for all $s \in \Omega$.*

The proof of this result relies on the fact that e^{-st} is analytic, but it is quite involved, so we omit it. A consequence of the theorem however, is that we may differentiate Laplace transforms to all orders. We use this fact in the next result.

Proposition 5.3. *Let f have Laplace transform $F(s)$. Then*

$$\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} F(s). \tag{5.8}$$

Proof. We write

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

Now differentiating under the integral sign gives

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt \tag{5.9}$$

$$= \int_0^\infty f(t)\frac{d}{ds}e^{-st}dt \tag{5.10}$$

$$= -\int_0^\infty tf(t)e^{-st} \tag{5.11}$$

$$= -\mathcal{L}(tf(t)). \tag{5.12}$$

Thus $\mathcal{L}(tf(t)) = -\frac{d}{ds}F(s)$. The general result now follows by induction. □

Let us recover our earlier result on the Laplace transform of t^n by this method.

Example 5.4. Find the Laplace transform of $f(t) = t$.
Solution. By the previous result we have

$$\mathcal{L}(f) = \int_0^\infty te^{-st}dt = -\frac{d}{ds}\int_0^\infty e^{-st}dt = -\frac{d}{ds}(\frac{1}{s}) = \frac{1}{s^2}. \tag{5.13}$$

Example 5.5. Find the Laplace transform of $f(t) = t^2$.
Solution. Using the previous proposition once more gives.

$$\mathcal{L}(f) = \int_0^\infty t^2e^{-st}dt = \frac{d^2}{ds^2}\int_0^\infty e^{-st}dt = (-1)^2\frac{d^2}{ds^2}(\frac{1}{s}) = \frac{2}{s^3}. \tag{5.14}$$

For general n we easily see by these calculations that

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}.$$

This gives us the result that we found previously.

Now let us present a few more commonly occurring Laplace transforms. The calculations are all straightforward.

Example 5.6. Find the Laplace transform of $f(t) = e^{-at}$.
Solution. This is again a simple integration.

$$\mathcal{L}(f) = \int_0^\infty e^{-at}e^{-st}dt = \left[\frac{1}{s+a}e^{-(a+s)t}\right]_0^\infty = \frac{1}{s+a}. \tag{5.15}$$

We apply this last result to simplify our calculations in the next examples.

Example 5.7. Find the Laplace transform of $f(t) = \sin(at)$.
Solution. To do this we observe that by Euler's formula

$$\sin(at) = \frac{1}{2i}(e^{iat} - e^{-iat}).$$

Hence

$$\begin{aligned}\mathcal{L}(f) = \frac{1}{2i}\mathcal{L}(e^{iat}) - \frac{1}{2i}\mathcal{L}(e^{-iat}) &= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right)\\ &= \frac{1}{2i}\left(\frac{s+ia-(s-ia)}{s^2+a^2}\right)\\ &= \frac{a}{s^2+a^2}.\end{aligned}$$

Example 5.8. Find the Laplace transform of $f(t) = \cos(at)$.
Solution. This is similar to the previous example. We have

$$\mathcal{L}(f) = \mathcal{L}\left(\frac{1}{2}(e^{iat} + e^{-iat})\right) = \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right) = \frac{s}{s^2+a^2}.$$

It is straightforward to compute Laplace transforms of a polynomial multiplied by a sine or cosine.

Example 5.9.

$$\mathcal{L}(t\cos 2t) = -\frac{d}{ds}\mathcal{L}(\cos 2t) = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{s^2-4}{(s^2+4)^2}. \tag{5.16}$$

A useful technique for computing Laplace transforms is to write the original function as a power series and integrate term by term.

Theorem 5.4. *Suppose that f is an integrable function which possesses a Laplace transform, and more over,*

$$f(t) = \sum_{n=0}^\infty \frac{a_n}{n!}t^n,$$

with the series converging absolutely. Then

$$\begin{aligned}\mathcal{L}(f)(s) &= \int_0^\infty \sum_{n=0}^\infty \frac{a_n}{n!} t^n e^{-st} dt \\ &= \sum_{n=0}^\infty \frac{a_n}{s^{n+1}}.\end{aligned}$$

Let us use this technique to calculate a Laplace transform.

Example 5.10. The Laplace transform of the zeroth order Bessel function $J_0(t)$ is

$$\mathcal{L}(J_0(t))(s) = \frac{1}{\sqrt{1+s^2}}. \tag{5.17}$$

We evaluated this Laplace transform in the section on Bessel functions. Here is another approach. We use the fact that

$$J_0(t) = \sum_{n=0}^\infty \frac{(-1)^n t^{2n}}{2^{2n} n! \Gamma(n+1)}. \tag{5.18}$$

Now we take the Laplace transform term by term.

$$\begin{aligned}\int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n t^{2n}}{2^{2n} n! \Gamma(n+1)} e^{-st} dt &= \sum_{n=0}^\infty \int_0^\infty \frac{(-1)^n t^{2n}}{2^{2n} n! \Gamma(n+1)} e^{-st} dt \\ &= \sum_{n=0}^\infty \frac{(-1)^n (2n)!}{2^{2n} n! \Gamma(n+1) s^{2n+1}}.\end{aligned}$$

Next we note that

$$\begin{aligned}\frac{1}{\sqrt{1+s^2}} &= \frac{1}{s\sqrt{1+\frac{1}{s^2}}} = \frac{1}{s}(1+\frac{1}{s^2})^{-\frac{1}{2}} \\ &= \frac{1}{s}(1 - \frac{1}{2}\frac{1}{s^2} + \frac{3}{8}\frac{1}{s^4} - \frac{5}{16}\frac{1}{s^6} + \cdots) \\ &= \sum_{n=0}^\infty \frac{(-1)^n (2n)!}{2^{2n} n! \Gamma(n+1) s^{2n+1}},\end{aligned}$$

which gives the desired result when we compare the two series.

Proposition 5.5. *Let f have Laplace transform F and assume that F is defined at $s+a$. Then*

$$\mathcal{L}(e^{-at} f(t)) = F(s+a).$$

Proof. By the definition of the Laplace transform we have

$$F(s+a) = \int_0^\infty f(t) e^{-(s+a)t} dt. \tag{5.19}$$

Hence

$$\mathcal{L}(e^{-at} f(t)) = F(s+a).$$

□

Example 5.11. We will use this to compute the Laplace transform of $f(t) = e^{-2t}\sin(4t)$. We have

$$\mathcal{L}(e^{-2t}\sin(4t))(s) = \frac{4}{(s+2)^2+16}. \tag{5.20}$$

Proposition 5.6. *Let $f(t)$ have Laplace transform $F(s)$. Then the Laplace transform of $f_a(t) = f(at)$ is*

$$\mathcal{L}(f_a)(s) = \frac{1}{a}F(\frac{s}{a}). \tag{5.21}$$

Proof. The proof of this result is again a simple computation with the integral. We have

$$\begin{aligned}\mathcal{L}(f_a)(s) &= \int_0^\infty f_a(t)e^{-st}dt \\ &= \int_0^\infty f(at)e^{-st}dt \\ &= \frac{1}{a}\int_0^\infty f(u)e^{-su/a}du = \frac{1}{a}F(\frac{s}{a}),\end{aligned} \tag{5.22}$$

where we used the substitution $at = u$ in the integral (5.22). □

Example 5.12. Let $f(t) = \sin t$. Then $\mathcal{L}(\sin t)(s) = \frac{1}{1+s^2}$. Therefore

$$\mathcal{L}(\sin(5t))(s) = \frac{1}{5}\frac{1}{1+s^2/5^2} = \frac{5}{s^2+25}.$$

Example 5.13. Find the Laplace transform of $J_0(4t)$.

By the proposition, we have

$$\mathcal{L}(J_0(4t))(s) = \frac{1}{4}\frac{1}{\sqrt{1+(s/4)^2}} = \frac{1}{\sqrt{16+s^2}}.$$

5.2. **Laplace Transforms of Derivatives.** The interest in Laplace transforms from the point of view of studying differential equations lies in the relationship between differentiation and Laplace transform. The next result is fundamental.

Proposition 5.7. *Let f be an n times differentiable function which has Laplace transform F. Then*

$$\mathcal{L}(f^{(n)}(x)) = -f^{(n-1)}(0) - sf^{(n-2)}(0) - \cdots - s^{n-1}f(0) + s^nF(s). \tag{5.23}$$

Proof. The definition of the Laplace transform gives

$$\begin{aligned}\mathcal{L}(f'(x)) = \int_0^\infty f'(x)e^{-sx}dx &= [f(x)e^{-sx}]_0^\infty + s\int_0^\infty f(x)e^{-sx}dx \\ &= -f(0) + sF(s),\end{aligned}$$

where $F(s)$ is the Laplace transform of f. Similarly

$$\begin{aligned}\mathcal{L}(f''(x)) = \mathcal{L}((f')') &= -f'(0) + s(-f(0) + sF(s)) \\ &= -f'(0) - sf(0) + s^2F(s).\end{aligned}$$

The result for $\mathcal{L}(f^{(n)})$ follows by induction. □

We give one further elementary property of Laplace transforms, which is extremely useful.

Proposition 5.8. *Let the Heaviside step function be defined by*

$$H(x-a) = \begin{cases} 1 & x \geq a \\ 0 & x < a \end{cases}$$

Then

$$\mathcal{L}(H(x-a)f(x-a)) = e^{-sa}F(s)$$

Proof. This is a simple calculation.

$$\begin{aligned}\mathcal{L}(H(x-a)f(x-a)) &= \int_0^\infty H(x-a)f(x-a)e^{-sx}dx \\ &= \int_a^\infty f(x-a)e^{-sx}dx, \quad t = x-a \\ &= \int_0^\infty f(t)e^{-s(t+a)}dt = e^{-sa}F(s).\end{aligned}$$

□

These results allow us to considerably extend the kinds of functions for which we can find the Laplace transform.

5.3. **The Inverse Laplace Transform.** One of the most important properties of Laplace transforms is that they are unique. This is encapsulated in the next result.

Theorem 5.9. *Suppose that $\mathcal{L}(f) = 0$. Then $f = 0$.*

Remark 5.10. If the student knows measure theory, the result of the preceding theorem really should be that $f = 0$ almost everywhere, but this will not concern us here.

Corollary 5.11. *Suppose that $\mathcal{L}(f) = \mathcal{L}(g)$. Then $f = g$.*

Proof. By Theorem 5.9, $\mathcal{L}(f-g) = 0$ implies that $f = g$. □

Because the Laplace transform is one to one, we can in principle go from a Laplace transform F to the original function f.

Definition 5.12. Suppose that $\mathcal{L}(f) = F$. That is, F is the Laplace transform of f. Then the inverse Laplace transform of F is the operator that recovers f from F. We write $\mathcal{L}^{-1}(F) = f$.

The inverse Laplace transform can actually be found in many cases by evaluating an integral. This will be discussed later. However in practice, we can find the inverse Laplace transform by using tables of known transforms and elementary properties of Laplace transforms. the most important is the following.

Theorem 5.13. *The inverse Laplace transform is linear. That is, for all constants a, b, we have*

$$\mathcal{L}^{-1}(aF + bG) = a\mathcal{L}^{-1}(F) + b\mathcal{L}^{-1}(G).$$

Proof. This is an exercise. □

We know that $\mathcal{L}(\cos(at)) = \dfrac{s}{s^2 + a^2}$, so that

$$\mathcal{L}^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos(at). \tag{5.24}$$

We also know the inverse Laplace transforms for many other elementary functions. Combining these, we can find the inverse Laplace transforms of many different functions.

Let us compute some examples.

Example 5.14. Find

$$\mathcal{L}^{-1}\left(\frac{s}{(s+2)(s^2+9)}\right).$$

We use partial fractions. We have

$$\frac{s}{(s+2)(s^2+9)} = \frac{A}{s+2} + \frac{Bs + C}{s^2+9}.$$

After some elementary algebra we obtain

$$\frac{s}{(s+2)(s^2+9)} = \frac{1}{13}\left(\frac{2s}{s^2+9} + \frac{9}{s^2+9} - \frac{2}{s+2}\right).$$

Thus by linearity of the inverse Laplace transform, we can write

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{s}{(s+2)(s^2+9)}\right) &= \frac{1}{13}\mathcal{L}^{-1}\left(\frac{2s}{s^2+9}\right) + \frac{1}{13}\mathcal{L}^{-1}\left(\frac{9}{s^2+9}\right) \\
&\quad - \frac{1}{13}\mathcal{L}^{-1}\left(\frac{2}{s+2}\right) \\
&= \frac{2}{13}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) + \frac{3}{13}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) \\
&\quad - \frac{2}{13}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) \\
&= \frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t) - \frac{2}{13}e^{-2t}.
\end{aligned}$$

Example 5.15. Calculate

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)(s^2+9)}\right).$$

The solution is again to use partial fraction decompositions. We write

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

and find after some algebra that $A = 1/5 = -C$ and $B = D = 0$. Thus

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{1}{5}\left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right).$$

From which

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)(s^2+9)}\right) &= \frac{1}{5}\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - \frac{1}{5}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) \\ &= \frac{1}{5}(\cos(2t) - \cos(3t)).\end{aligned}$$

Often we need to use some of the elementary properties of Laplace transforms to find the inverse transform. Here are two simple illustrations.

Example 5.16. Find

$$\mathcal{L}^{-1}\left(\frac{s+4}{((s+4)^2+4)((s+4)^2+9)}\right).$$

We know that $\mathcal{L}^{-1}(F(s+a)) = e^{-at}\mathcal{L}^{-1}(F(s))$ by Proposition 5.5. So by the previous example we have

$$\mathcal{L}^{-1}\left(\frac{s+4}{((s+4)^2+4)((s+4)^2+9)}\right) = \frac{1}{5}e^{-4t}(\cos(2t) - \cos(3t)).$$

Example 5.17. Find $\mathcal{L}^{-1}(e^{-3s}/s^4)$, where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

Solution. We know that $\frac{1}{6}\mathcal{L}(x^3) = \frac{1}{6}\frac{3!}{s^4} = \frac{1}{s^4}$. So by Proposition 5.8 we have

$$e^{-3s}\frac{1}{s^4} = \mathcal{L}(H(x-3)\frac{1}{6}(x-3)^3).$$

Hence

$$\begin{aligned}\mathcal{L}^{-1}(e^{-3s}/s^4) &= \frac{1}{6}(x-3)^3 H(x-3) \\ &= \begin{cases} \frac{1}{6}(x-3)^3, & x > 3 \\ 0, & x \leq 3. \end{cases}\end{aligned}$$

5.4. **Inversion by series.** This method is quite useful and actually quite straightforward. Let us consider the Laplace transform

$$F(s) = \frac{s}{s^2+a^2} = \frac{1}{s(1+(a/s)^2)}. \tag{5.25}$$

We write this as the sum of a geometric series.

$$\frac{1}{s(1+(a/s)^2)} = \frac{1}{s}\left(1 - \frac{a^2}{s^2} + \frac{a^4}{s^4} - \frac{a^6}{s^6} + \cdots\right). \tag{5.26}$$

Inverting term by term, according to the rule

$$\mathcal{L}^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{1}{n!}t^n, \tag{5.27}$$

we have

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) &= 1 - \frac{(at)^2}{2!} + \frac{(at)^4}{4!} - \frac{(at)^6}{6!} + \cdots \\ &= \cos(at).\end{aligned}$$

Example 5.18. We invert the Laplace transform $F(s) = \sin(a/s)$. Expanding in a series we get

$$\sin\left(\frac{a}{s}\right) = \frac{a}{s} - \frac{a^3}{3!s^3} + \cdots \tag{5.28}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{a^{2n+1}}{(2n+1)!s^{2n+1}} \tag{5.29}$$

Inverting term by term gives

$$f(t) = \mathcal{L}^{-1}(\sin(a/s)) = \sum_{n=0}^{\infty} \frac{(-1)^n (at)^{2n}}{(2n)!(2n+1)!}.$$

This can be reduced to an expression involving Bessel functions, but we will not consider this. We will however do a similar example later. The point is that even if we do not recognise the function, we still have a series for the inverse Laplace transform which we can use.

Example 5.19. Recall that for $|x| < 1$ the inverse tan can be expressed as

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots .$$

From this we have

$$\tan^{-1}\left(\frac{a}{s}\right) = \frac{a}{s} - \frac{a^3}{3s^3} + \frac{a^5}{5s^5} - \cdots \tag{5.30}$$

This means that

$$\begin{aligned}\mathcal{L}^{-1}\left(\tan^{-1}\left(\frac{a}{s}\right)\right) &= a - \frac{a^3t^2}{3!} + \frac{a^5t^4}{5!} - \cdots \\ &= \frac{1}{t}\left(at - \frac{1}{3!}(at)^3 + \frac{1}{5!}(at)^5 - \cdots\right) \\ &= \frac{\sin(at)}{t}.\end{aligned}$$

We might like to verify this Laplace transform directly by computing

$$F(s) = \int_0^\infty \frac{\sin(at)}{t} e^{-st} dt.$$

The easiest way to do this is to differentiate under the integral sign. We find that

$$\begin{aligned}F'(s) &= \frac{d}{ds}\int_0^\infty \frac{\sin(at)}{t} e^{-st} dt \\ &= \int_0^\infty \frac{\partial}{\partial s}\frac{\sin(at)}{t} e^{-st} dt \\ &= -\int_0^\infty \sin(at) e^{-st} dt \\ &= -\frac{a}{s^2+a^2}.\end{aligned}$$

Integration then gives us

$$F(s) = -\tan^{-1}\left(\frac{s}{a}\right) + C.$$

C is a constant of integration. Now

$$F(0) = \int_0^\infty \frac{\sin(at)}{t} dt = \frac{\pi}{2}.$$

(We have used a known integral). This gives us

$$F(s) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right). \tag{5.31}$$

This seems to be different from our previous answer. But let us investigate further. We know that

$$\frac{\pi}{2} - \tan^{-1} n = \lim_{m\to\infty}\left(\tan^{-1} m - \tan^{-1} n\right) = \lim_{m\to\infty} \tan^{-1}\left(\frac{m-n}{1+mn}\right). \tag{5.32}$$

With $n = s/a$ we have

$$\begin{aligned}\tan^{-1} m - \tan^{-1}\left(\frac{s}{a}\right) &= \tan^{-1}\left(\frac{m - s/a}{1+ms/a}\right) \\ &= \tan^{-1}\left(\frac{1-s/(am)}{1/m + s/a}\right).\end{aligned}$$

Now

$$\lim_{m\to\infty} \tan^{-1}\left(\frac{1-s/(am)}{1/m+s/a}\right) = \tan^{-1}\left(\frac{a}{s}\right). \tag{5.33}$$

So that

$$\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right). \tag{5.34}$$

and there is no contradiction between the two answers.

Example 5.20. We now invert $F(s) = \ln\left(\dfrac{s+a}{s+b}\right)$. We know that $\ln\left(\frac{s+a}{s+b}\right) = \ln(s+a) - \ln(s+b)$ and $\int \frac{ds}{s+a} = \ln(s+a)$. Using

$$\frac{1}{s+a} = \frac{1}{s(1+a/s)} = \frac{1}{s}\left(1 - \frac{a}{s} + \left(\frac{a}{s}\right)^2 - \left(\frac{a}{s}\right)^3 + \cdots\right)$$

we have

$$\ln\left(\frac{s+a}{s+b}\right) = -\frac{b-a}{s} + \frac{b^2-a^2}{2s^2} - \frac{(b^2-a^3)}{3s^3} + \cdots. \tag{5.35}$$

Inverting term by term gives

$$\begin{aligned}
f(t) = \mathcal{L}^{-1}\left(\ln\left(\frac{s+a}{s+b}\right)\right) &= -(b-a) + \frac{b^2-a^2}{2!}t - \frac{(b^3-a^3)}{3!}t^2 + \cdots \\
&= \frac{1}{t}\left(\left(1 - bt + \frac{b^2t^2}{2!} - \frac{b^3t^3}{3!} + \cdots\right) - \left(1 - at + \frac{a^2t^2}{2!} - \frac{a^3t^3}{3!} + \cdots\right)\right) \\
&= \frac{1}{t}(e^{-bt} - e^{-at}).
\end{aligned}$$

5.5. **Convolution.** Convolution is the "right" multiplication for an integral transform. It is obvious that $\mathcal{L}fg \neq \mathcal{L}(f)\mathcal{L}(g)$. However this begs the question, what is the inverse Laplace transform of $F(s)G(s)$ in terms of the inverse Laplace transforms of F and G? Is there a simple way of working this out? It turns out that the answer is yes.

We proceed as follows. Let $F = \mathcal{L}(f), G = \mathcal{L}(g)$, then

$$\begin{aligned}
F(s)G(s) &= \int_0^\infty e^{-sx} f(x)dx \int_0^\infty e^{-sy} g(y)dy \\
&= \int_0^\infty \int_0^\infty e^{-s(x+y)} f(x)g(y)dydx
\end{aligned}$$

Now let $x = t - u$ and $u = y$ so that $t = x + y$. The double integral then becomes,

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_0^t e^{-st} f(t-u)g(u)dudt \\ &= \int_0^\infty e^{-st} \left(\int_0^t f(t-u)g(u)du \right) dt \\ &= \mathcal{L}\left(\int_0^t f(t-u)g(u)du \right). \end{aligned}$$

Hence

$$\int_0^x f(x-u)g(u)du = \mathcal{L}^{-1}(FG).$$

This gives us the desired inverse Laplace transform in terms of the *convolution* for the Laplace transform.

Definition 5.14. Let f and g be two integrable functions. Then the convolution for the Laplace transform is given by

$$(f * g)(t) = \int_0^t f(t-u)g(u)du. \tag{5.36}$$

We have already proved the next result.

Theorem 5.15. *Let f and g be functions which are integrable and possess Laplace transforms. Then $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$.*

As an application of this result we do the following.

Example 5.21. Find the inverse Laplace transform of

$$H(s) = \frac{1}{s^2(s^2+1)}.$$

Solution. Let $F(s) = 1/s^2$ and $G(s) = 1/(s^2+1)$. Then we know that $f(x) = \mathcal{L}^{-1}(F(s)) = t$ and $g(t) = \mathcal{L}^{-1}(G(s)) = \sin t$. Hence

$$\begin{aligned} \mathcal{L}^{-1}(FG) &= \int_0^t f(t-u)g(u)du \\ &= \int_0^t (t-u)\sin u du \\ &= [-(t-u)\cos u - \sin u]_{u=0}^{u=t} \\ &= t - \sin t \end{aligned}$$

Example 5.22. Compute the Laplace transform of the integral $\int_0^t f(x)dx$.

Solution We use the convolution theorem. So that

$$\begin{aligned}\mathcal{L}\left(\int_0^t f(x)dx\right) &= \mathcal{L}(f * 1)\\ &= \mathcal{L}(1)\mathcal{L}(f)\\ &= \frac{1}{s}F(s),\end{aligned}$$

where F is the Laplace transform of f.

5.6. **Solving Differential Equations.** Constant coefficient ordinary differential equations can be solved by Laplace transform, relatively easily. Let us consider two simple examples.

Example 5.23. We solve the equation

$$y'' + 5y' + 6y = t^2,$$

subject to $y(0) = 1, y'(0) = 2$. The idea is to take the Laplace transform of both sides of the equation, so that we have

$$\mathcal{L}\left(y'' + 5y' + 6y\right) = \mathcal{L}\left(t^2\right). \tag{5.37}$$

Using our results for the Laplace transforms of the first and second derivatives of a function, we can write

$$(s^2 + 5s + 6)Y(s) - (5 + s)y(0) - y'(0) = \frac{2}{s^3}, \tag{5.38}$$

where Y is the Laplace transform of y. Rearranging this gives

$$Y(s) = \frac{7+s}{s^2+5s+6} + \frac{2}{s^3(s^2+5s+6)}. \tag{5.39}$$

Using partial fractions we obtain

$$Y(s) = \frac{19}{4(s+2)} - \frac{106}{27(s+3)} + \frac{19}{108s} - \frac{5}{18s^2} + \frac{1}{3s^3}. \tag{5.40}$$

We recover the solution by taking the inverse Laplace transform of $Y(s)$. So that

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left(Y(s)\right)\\ &= \mathcal{L}^{-1}\left(\frac{19}{4(s+2)} - \frac{106}{27(s+3)} + \frac{19}{108s} - \frac{5}{18s^2} + \frac{1}{3s^3}\right)\\ &= \frac{t^2}{6} - \frac{5t}{18} - \frac{106e^{-3t}}{27} + \frac{19e^{-2t}}{4} + \frac{19}{108}.\end{aligned}$$

We next consider a problem where the right hand side of the equation is left unspecified.

Example 5.24. Suppose that we want to solve the equation

$$y'' + a^2 y = f(t),$$

subject to the condition $y(0) = 0, y'(0) = 1$. We take the Laplace transform of both sides, to obtain

$$(s^2 + a^2)Y(s) - sy(0) - y'(0) = (s^2 + a^2)Y(s) - 1 = F(s), \tag{5.41}$$

where Y is the Laplace transform of y and F is the Laplace transform of f. Consequently

$$Y(s) = \frac{F(s)}{s^2 + a^2} + \frac{1}{s^2 + a^2} \tag{5.42}$$

Now $\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin(at)$. By the Convolution Theorem we can also write

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s^2 + a^2}\right) = \frac{1}{a}\int_0^t f(t-u)\sin(au)du. \tag{5.43}$$

From which we have the solution

$$y(t) = \frac{1}{a}\int_0^t f(t-u)\sin(au)du + \frac{1}{a}\sin(at). \tag{5.44}$$

It is possible to solve some nonconstant coefficient differential equations using the Laplace transform. We provide some examples.

Example 5.25. Solve the ODE $ty'' + y = 0, y(0) = 0$. We let $Y(s) = \mathcal{L}(y)$ and then we have

$$\begin{aligned}\mathcal{L}(ty'') &= -\frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) \\ &= -s^2Y'(s) - 2sY(s).\end{aligned}$$

The equation then becomes

$$s^2\frac{dY}{ds} + (2s-1)Y(s) = 0. \tag{5.45}$$

The solution of this first order ODE is

$$Y(s) = \frac{A}{s^2}e^{-1/s}. \tag{5.46}$$

We have to invert this Laplace transform. We do it term by term. We are here using a result presented previously. Note that

$$\frac{1}{s^2}e^{-1/s} = \frac{1}{s^2}(1 - \frac{1}{s} + \frac{1}{2!s^2} - \frac{1}{3!s^3} + \frac{1}{4!s^4} - \cdots. \tag{5.47}$$

Now we use the fact that $\mathcal{L}^{-1}(1/s^{n+1}) = \frac{1}{n!}t^n$. This gives

$$\begin{aligned}\mathcal{L}^{-1}[\frac{1}{s^2}e^{-1/s}] &= \mathcal{L}^{-1}[1/s^2] - \mathcal{L}^{-1}[1/s^3] + \frac{1}{2!}\mathcal{L}^{-1}[1/s^4] - \cdots \\ &= t - \frac{t^2}{2!} + \frac{t^3}{2!3!} - \frac{t^4}{3!4!} + \cdots \\ &= t\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!(n+1)!}.\end{aligned}$$

To identify this function, we note that

$$J_a(t) = \left(\frac{t}{2}\right)^a \sum_{n=0}^{\infty} \frac{(-1)^n (t/2)^{2n}}{n!\Gamma(n+a+1)}. \tag{5.48}$$

Let us take $a = 1$. Then we see that

$$J_1(2\sqrt{t}) = \sqrt{t} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!(n+1)!}. \tag{5.49}$$

From this we conclude that

$$\mathcal{L}^{-1}[\frac{1}{s^2} e^{-1/s}] = \sqrt{t} J_1(2\sqrt{t}). \tag{5.50}$$

Thus the solution of the ODE turns out to be $y(t) = A\sqrt{t} J_1(2\sqrt{t})$.

Remark 5.16. Using the technique of the previous question, we can show that for $n > 0$ and $a > 0$

$$\mathcal{L}^{-1}\left[\frac{1}{s^n} e^{-\frac{a}{s}}\right] = \left(\frac{a}{t}\right)^{\frac{1-n}{2}} J_{n-1}\left(2\sqrt{at}\right). \tag{5.51}$$

Example 5.26. Solve the equation $xy'' + (b - x)y' - ay = 0$. This is called the confluent hypergeometric equation. We take a slightly different approach to this problem. We suppose that the solution can be written

$$y(x) = \int_0^\infty h(t) e^{-xt} dt.$$

This method assumes that $t^2 h(t) e^{-xt} \to 0$ at $t \to \infty$, in order for the solution to be expressible in this form. If there is a solution which does not have this property, then it will not be expressible as a Laplace transform.

Now to calculate y'' we differentiate the exponential under the integral sign, to get $y''(x) = \int_0^\infty t^2 h(t) e^{-xt} dt$. Then using integration by parts we have

$$\begin{aligned} xy''(x) &= \int_0^\infty x t^2 h(t) e^{-xt} dt \\ &= -\int_0^\infty t^2 h(t) \frac{d}{dt} e^{-xt} dt \\ &= -\left[t^2 h(t) e^{-xt}\right]_0^\infty + \int_0^\infty \frac{d}{dt}(t^2 h(t)) e^{-xt} dt \\ &= \int_0^\infty (t^2 h'(t) + 2t h(t)) e^{-xt} dt. \end{aligned}$$

Next we have

$$\begin{aligned}(b-x)y'(x) &= -(b-x)\int_0^\infty th(t)e^{-xt}dt \\ &= -b\int_0^\infty th(t)e^{-xt}dt - \int_0^\infty th(t)\frac{d}{dt}e^{-xt}dt \\ &= -b\int_0^\infty th(t)e^{-xt}dt - \left[th(t)e^{-xt}\right]_0^\infty \\ &\quad + \int_0^\infty (th'(t)+h(t))e^{-xt}dt \\ &= \int_0^\infty (th'(t)+(1-bt)h(t))e^{-xt}dt.\end{aligned}$$

Combing we have

$$\begin{aligned}&xy''+(b-x)y'-ay = \\ &\int_0^\infty \left(t^2h'(t)+2th(t)+th'(t)+(1-bt)h(t)-ah(t)\right)e^{-xt}dt \\ &= \int_0^\infty (t(1+t)h'(t)+(1-a+(2-b)t)h(t))e^{-xt}dt = 0.\end{aligned}$$

In order for this to hold, the integrand must be zero. So we require

$$t(1+t)h'(t)+(1-a+(2-b)t)h(t)=0. \tag{5.52}$$

This is a first order separable ODE

$$h'(t)+\frac{1-a+(2-b)t}{t(1+t)}h(t)=0. \tag{5.53}$$

So

$$\int \frac{dh}{h} = -\int \frac{1-a+(2-b)t}{t(1+t)}dt \tag{5.54}$$

or

$$\ln h(t) = (b-a-1)\ln(t+1)+(a-1)\ln(t)+C. \tag{5.55}$$

Hence

$$h(t) = At^{a-1}(1+t)^{b-a-1}. \tag{5.56}$$

Thus the solution is

$$h(t) = A\int_0^\infty t^{a-1}(1+t)^{b-a-1}e^{-xt}dt. \tag{5.57}$$

The most common choice for the constant is $A = 1/\Gamma(a)$. Then

$$U(a,b,x) = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{b-a-1}e^{-xt}dt \tag{5.58}$$

is the confluent hypergeometric function of the second kind, or Tricomi's confluent hypergeometric function.

There is another solution of this equation, Kummer's confluent hypergeometric function which can be represented as

$$ {}_1F_1(a,b,x) = \sum_{k=0}^{\infty} \frac{(a)_n}{k!(b)_n} x^n, \tag{5.59} $$

where $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$. This can be found by the method of Frobenius. It was not found by the Laplace transform method because of the convergence condition we mentioned before. For large x, ${}_1F_1(a,b,x)$ grows like $\frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b}$.

It is possible to establish an integral representation for Kummer's function. Specifically

$$ {}_1F_1(a,b,x) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{xt} t^{a-1}(1-t)^{b-a-1} dt. \tag{5.60} $$

The general hypergeometric function is

$$ {}_pF_q(a_1,...a_p;b_1,...,b_q,x) = \sum_{k=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{k!(b_1)_n \cdots (b_q)_n} x^n, \tag{5.61} $$

and the series converges if $p \leq q$. These satisfy higher order differential equations. We obviously require that the b's are not negative integers, to avoid dividing by zero. For example

$$ (-2)_3 = -2 \times (-1) \times (0) = 0. $$

It should also be clear that if one of the a's is a negative integer, then the series will terminate, and so the corresponding hypergeometric function is a polynomial.

The importance of the hypergeometric functions lies in their connection with other functions. A very large class of differential equations have solutions which can be expresses as hypergeometric functions of some type. For example it is clear that if $a = b$, then

$$ {}_1F_1(a,b,x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^n = e^x. \tag{5.62} $$

There are many such relations. Some are not obvious, such as

$$ {}_1F_1(\alpha+1/2, 2\alpha+1, 2ix) = \Gamma(\alpha+1) e^{ix} (x/2)^{-\alpha} J_\alpha(x). \tag{5.63} $$

So Bessel functions can be expressed in terms of hypergeometric functions. Many functions familiar to you, such as polynomials, logarithms, sines and cosines, the error function from probability, Bessel functions, Legendre functions, Hermite functions, etc are expressible as hypergeometric functions.

5.7. **Solving PDEs with Laplace Transform.** The differential equations that we have solved so far are of course all solvable by variation of parameters or series methods. It turns out that the Laplace transform is actually very useful for the solution of partial differential equations than ordinary differential equations. The idea is to turn a PDE into an ODE. We will discuss PDEs in more detail in the final section of the course. Here we simply give two fairly typical examples.

Example 5.27. Solve the first order PDE,

$$x\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x \quad x, t > 0 \tag{5.64}$$

subject to the conditions

$$u(x, 0) = 0, \quad u(0, t) = 0.$$

Solution. We take the Laplace transform in the t variable.

$$\mathcal{L}\left(x\frac{\partial u}{\partial t}\right) = x\int_0^\infty \frac{\partial u}{\partial t}e^{-st} = x(-u(x, 0) + s\bar{U}(x, s)),$$

where $\bar{U}(x, s) = \int_0^\infty u(x, t)e^{-st}dt$. Also.

$$\mathcal{L}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\mathcal{L}(u) = \frac{\partial}{\partial x}\bar{U}(x, s),$$

and x is treated as a constant. Hence we have,

$$-xu(x, 0) + xs\bar{U}(x, s) + \frac{\partial \bar{U}}{\partial x}(x, s) = \frac{x}{s}.$$

So

$$\frac{d}{dx}\bar{U}(x, s) + xs\bar{U}(x, s) = \frac{x}{s}.$$

We solve this ODE by multiplying through by the integrating factor of $e^{sx^2/2}$. This gives

$$e^{s\frac{x^2}{2}}\frac{d\bar{U}}{dx} + sxe^{s\frac{x^2}{2}}\bar{U} = \frac{x}{s}e^{s\frac{x^2}{2}}.$$

Hence,

$$\frac{d}{dx}(e^{s\frac{x^2}{2}}\bar{U}) = \frac{x}{s}e^{s\frac{x^2}{2}}.$$

Therefore

$$e^{s\frac{x^2}{2}}\bar{U} = \int \frac{x}{s}e^{s\frac{x^2}{2}}dx = \frac{1}{s^2}e^{s\frac{x^2}{2}} + C,$$

for some constant of integration C. Solving for $\bar{U}(x, s)$ gives,

$$\bar{U}(x, s) = \frac{1}{s^2} + Ce^{-s\frac{x^2}{2}}.$$

Now we need to determine C. To do this we take the Laplace transform of the initial condition to get,

$$\bar{U}(0, s) = \mathcal{L}(u(0, t)) = \mathcal{L}(0) = 0.$$

We can now find our constant C. We have

$$\bar{U}(0,s) = \frac{1}{s^2} + C = 0 \Rightarrow C = -\frac{1}{s^2}.$$

Hence

$$\bar{U}(x,s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s\frac{x^2}{2}}$$

To find $u(x,t)$ we now take the inverse Laplace transform.

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1}{s^2} - \frac{1}{s^2}e^{-s\frac{x^2}{2}}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2}e^{-s\frac{x^2}{2}}\right)\\ &= t - \mathcal{L}^{-1}\left(\frac{1}{s^2}e^{-s\frac{x^2}{2}}\right).\end{aligned}$$

Let $a = x^2/2$, now $\mathcal{L}(y(t-a)H(t-a) = e^{-sa}\mathcal{L}(y))$ so,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}e^{-s\frac{x^2}{2}}\right) = (t - \frac{1}{2}x^2)H(t - \frac{1}{2}x^2).$$

Consequently the solution of the PDE is

$$\begin{aligned}u(x,t) &= t - (t - \frac{1}{2}x^2)H(t - \frac{1}{2}x^2)\\ &= \begin{cases}\frac{1}{2}x^2 & t \geq \frac{1}{2}x^2\\ t & t < \frac{1}{2}x^2.\end{cases}\end{aligned}$$

The Laplace transform is often used to solve equations of the form

$$u_t = a(x)u_{xx} + b(x)u_x + c(x)u, \tag{5.65}$$

by taking the Laplace transform in the t variable and producing an ODE, which can be solved by the methods that we have already developed. Let us consider an example of such a problem.

Example 5.28. We want to solve the following initial-boundary value problem for the heat equation.

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq a,\ t \geq 0$$

$$u(x,0) = 0, \quad \left(\frac{\partial u}{\partial x}\right)_{x=a} = 0, \quad u(0,t) = u_0$$

The heat equation will be discussed in greater detail in the next chapter. *Solution.* We take the Laplace transform in the t variable. This leads to the following ODE for the Laplace transform of u.

$$\begin{aligned}\mathcal{L}\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{d^2}{dx^2}\bar{U} &= -\frac{1}{k}u(x,0) + \frac{s}{k}\bar{U}(x,s)\\ &= \frac{s}{k}\bar{U}(x,s)\end{aligned}$$

This is a second order constant coefficient ODE for $\bar{U}$. The solution is of the form

$$\bar{U}(x,s) = c_1 e^{x\sqrt{\frac{s}{k}}} + c_2 e^{-x\sqrt{\frac{s}{k}}}.$$

However it is more convenient to write the solution as,

$$\bar{U}(x,s) = A\cosh\left(x\sqrt{\frac{s}{k}}\right) + B\sinh\left(x\sqrt{\frac{s}{k}}\right).$$

Now $\mathcal{L}(u(0,t)) = \mathcal{L}(u_0) = u_0/s = \bar{U}(0,s)$. Also,

$$\left(\frac{\partial \bar{U}}{\partial x}\right)_{x=a} = \int_0^\infty \left(\frac{\partial u}{\partial x}\right)_{x=a} e^{-st}dt = 0$$

So $\bar{U}(0,s) = A, u_0/s$ and

$$\left(\frac{\partial \bar{U}}{\partial x}\right)_{x=a} = A\sqrt{\frac{s}{k}}\sinh\left(a\sqrt{\frac{s}{k}}\right) + B\sqrt{\frac{s}{k}}\cosh\left(a\sqrt{\frac{s}{k}}\right) = 0$$

$$B = -\frac{u_0\sinh\left(a\sqrt{\frac{s}{k}}\right)}{s\cosh\left(a\sqrt{\frac{s}{k}}\right)}.$$

Now we use various identities for the hyperbolic functions to obtain

$$\begin{aligned}
\bar{U}(x,s) &= \frac{u_0}{s}\left(\cosh\left(x\sqrt{\frac{s}{k}}\right) - \frac{\sinh\left(a\sqrt{\frac{s}{k}}\right)}{\cosh\left(a\sqrt{\frac{s}{k}}\right)}\sinh\left(x\sqrt{\frac{s}{k}}\right)\right)\\
&= \frac{u_0}{s}\left(\frac{\cosh\left(a\sqrt{\frac{s}{k}}\right)\cosh\left(x\sqrt{\frac{s}{k}}\right) - \sinh\left(a\sqrt{\frac{s}{k}}\right)\sinh\left(x\sqrt{\frac{s}{k}}\right)}{\cosh\left(a\sqrt{\frac{s}{k}}\right)}\right)\\
&= \frac{u_0}{s}\frac{\cosh\left((x-a)\sqrt{\frac{s}{k}}\right)}{\cosh a\sqrt{\frac{s}{k}}}.
\end{aligned}$$

In order to determine u we have to invert this Laplace transform. This is accomplished with the aid of tables of transforms. It turns out that the inverse Laplace transform is given by an infinite series. We have

$$\begin{aligned}
u(x,t) &= \mathcal{L}^{-1}\left(\frac{u_0}{s}\frac{\cosh\left((x-a)\sqrt{\frac{s}{k}}\right)}{\cosh a\sqrt{\frac{s}{k}}}\right)\\
&= u_0\left[1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}e^{-(2n-1)^2\pi^2 kt/4a^2}\sin\left(\frac{2n-1}{2a}\right)\pi x\right].
\end{aligned}$$

The solution that we have obtained here is an example of a Fourier series. It is possible to derive it using a technique called separation of variables, which we will study in the next chapter.

5.8. **Deeper Properties of Laplace Transforms.** An important problem is to know exactly when a given function $F(s)$ is the Laplace transform of some function f. The next result gives a partial answer to that question. We state the result without proof. The conditions may be found in a number of texts. Here we have taken them from the book by Zayed, [5].

Definition 5.17. A function $f : X \to \mathbb{R}$ is locally integrable if for every finite interval $I \subseteq X$, the integral $\int_I f(t)dt$ exists.

Theorem 5.18. *A function F is the Laplace transform of some locally integrable function f if any of the following conditions is satisfied.*

(1) *F is analytic in some half plane $Re(s) > \sigma_c$ with*

$$\lim_{|\tau|\longrightarrow\infty} |\tau||F(\sigma + i\tau)| = 0, \quad for\ \sigma > \sigma_c$$

(2) *F is a rational function of the form $F(s) = P(s)/Q(s)$ where P and Q are polynomials with deg $Q >$ deg P.*
(3) *$F(s) = G(s)H(s)$ where G and H are Laplace transforms of locally integrable functions.*
(4) *$F(s) = G(s)e^{-sT}$ $T > 0$ where G is the Laplace transform of a locally integrable function g.*

Proof. We will not go into details for the proof of part 1, but the remaining parts are quite straightforward. For part 2, the proof follows from the fact that every such function has a partial fraction decomposition. We can then invert the Laplace transform term by term. For part 3, the inverse Laplace transform will be the convolution of f and g. Finally, for part 4, the inverse Laplace transform will be $H(t - T)g(t - T)$. □

One of the most important properties of the Laplace transform is that a Laplace transform is analytic. This result was quoted at the beginning of the notes. In fact the following deeper result from [5] holds. We have already used this result to invert several Laplace transforms.

Theorem 5.19. *The Laplace transform $F(s)$ of a locally integrable function f is an entire function in $1/s$ if and only if f is an entire function of order 1 and of minimal type, that is f is analytic for all t and*

$$f(t) = O(e^{-\varepsilon|t|}), \quad as\ t \longrightarrow \infty.$$

Moreover if

$$F(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}}, \quad |s| > C > 0,$$

then $f(t)$ is an entire function of exponential type α (that is, $f(t) = O(e^{\alpha|t|})$ as $t \longrightarrow \infty$) and

$$f(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}.$$

To see why this works, we let $f(t) = \displaystyle\sum_{n=0}^{\infty} \frac{a_n t^n}{n!}$ and compute the Laplace transform, assuming it is valid to interchange sum and integral. Thus

$$\begin{aligned}
\mathcal{L}(f) &= \int_0^\infty \sum_{n=0}^{\infty} \frac{a_n t^n}{n!} e^{-st} dt \\
&= \sum_{n=0}^{\infty} \frac{a_n}{n!} \int_0^\infty t^n e^{-st} dt \\
&= \sum_{n=0}^{\infty} \frac{a_n}{n!} \frac{n!}{s^{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}} = F(s).
\end{aligned}$$

A little more is needed to prove the theorem, but the idea is relatively simple.

To conclude our discussion of the Laplace transform we present the best known inversion theorem. There are in fact several inversion theorems for the Laplace transform, but this is the most useful. Before presenting the result, we need a definition.

Definition 5.20. A function f is said to have bounded variation on $[a, b]$ if for every partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ the quantity

$$\text{variation}(f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

is bounded.

Theorem 5.21 (Laplace Transform Inversion)**.** *If $f(t)$ is a locally integrable function on $[0, \infty)$ such that,*

1. *f is of bounded variation in a neighborhood of a point $t_0 \geq 0$ (a right hand neighborhood if $t_0 = 0$).*
2. *The integral $\int_0^\infty f(t)e^{-st}dt$ converges absolutely on $Re(s) = c$ then,*

$$\lim_{T \longrightarrow \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) e^{st} ds = \begin{cases} 0 & t_0 < 0 \\ f(0^+)/2 & t_0 = 0 \\ \frac{1}{2}(f(t_0^+) + f(t_0^-)) & t_0 > 0. \end{cases}$$

In particular if f is differentiable on $(0,\infty)$ and satisfies (1) and (2) then,

$$\lim_{T\longrightarrow\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}F(s)e^{st}ds=f(t)\quad 0<t<\infty.$$

Proof. This result can be established from the Fourier inversion theorem, but we will not discuss it. □

Remark 5.22. The integral is taken in the principal value sense since in general $\int_{c-i\infty}^{c+i\infty}F(s)e^{st}ds$ does not converge.

Example 5.29. $F(s)=1/(s^2+1)$ there are poles at $s=\pm i$. So that

$$\mathit{Residue}\left(\frac{e^{st}}{s^2+1},s=i\right)=\lim_{s\longrightarrow i}\left(\frac{(s-i)e^{st}}{(s-i)(s+i)}\right)=\frac{1}{2i}e^{it}$$

$$\mathit{Residue}\left(\frac{e^{st}}{s^2+1},s=-i\right)=\lim_{s\longrightarrow -i}\left(\frac{(s+i)e^{st}}{(s-i)(s+i)}\right)=-\frac{1}{2i}e^{-it}$$

By the Cauchy residue theorem we then have

$$\int_{c-i\infty}^{c+i\infty}e^{st}F(s)ds=2\pi i\sum \mathit{Residues}$$

$$=2\pi i(\frac{1}{2i}(e^{it}-e^{-it}))=2\pi i\sin t$$

So

$$\mathcal{L}^{-1}(\frac{1}{s^2+1})=\frac{1}{2\pi i}2\pi i\sin t=\sin t$$

Our last result is the Laplace transform version of a result known as the Plancherel Theorem which holds for the Fourier transform.

Theorem 5.23 (Parseval's Theorem)**.** *If F is the Laplace transform of a function $f\in L^2(0,\infty)$ then,*

$$\lim_{\sigma\longrightarrow 0^+}F(\sigma+it)=F(it), \tag{5.66}$$

$$\lim_{T\longrightarrow\infty}\frac{1}{2\pi i}\int_{-iT}^{iT}F(s)e^{st}ds=\begin{cases}f(t) & t\geq 0\\ 0 & t<0\end{cases} \tag{5.67}$$

and

$$\int_0^\infty e^{-2ct}|f(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(c+i\tau)|^2d\tau,$$

where $c>0$ and the limits are taken in the L^2 sense.

6. Fourier Series and Separation of Variables

6.1. **The Heat Equation.** In the eighteenth century many questions arose about the ways in which a function can be represented by simpler functions, such as polynomials and trigonometric functions. We have seen that it is possible to represent a particular class of functions as power series. Functions which can be expressed as power series are known as the analytic functions and the technique of expanding a function in a Taylor series has been crucial to an enormous amount of mathematics over the centuries.

The study of partial differential equations however led to the consideration of other kinds of expansions for functions. The most important of these relates to trigonometric series. Three partial differential equations arise in numerous areas of mathematics and its applications, particularly in physics. These are the heat, wave and Laplace equations. There are many more PDEs which arise in applications, but these three are the most important.

The n dimensional heat equation is

$$\frac{1}{k}\frac{\partial u}{\partial t} = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}. \tag{6.1}$$

It models the behaviour of heat in a solid body. It is an example of a diffusion equation, because it describes how heat diffuses through a body over time. The constant k depends on the medium in which the heat is being conducted. You can think of a solution $u(x, y, z, t)$ as giving the temperature at the point (x, y, z) at a time t. Study of the heat equation is of enormous importance in many problems in physics and engineering.

The wave equation in n dimensions is

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}. \tag{6.2}$$

This equation describes wave motion, where c is the speed of the wave. It is important in the theory of sound, electromagnetism, fluid mechanics and many other fields.

Finally, Laplace's equation in n dimensions is the equation

$$\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0. \tag{6.3}$$

It arises in numerous areas from the study of complex variables, to Newton's theory of gravity.

Partial Differential Equations are classified into different types. For a linear PDE in two dimensions, the classification is as follows.

Definition 6.1. A partial differential equation

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0$$

is said to be

(i) Elliptic if $ac - b^2 > 0$

(ii) Parabolic if $ac - b^2 = 0$

(iii) Hyperbolic if $ac - b^2 < 0$.

In higher dimensions there is a similar definition. The heat equation is parabolic, the wave equation is hyperbolic and the Laplace equation is elliptic. The classification comes from the classification of curves in the plane. A curve of the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

is an ellipse if $ac - b^2 > 0$, a parabola if $ac - b^2 = 0$ and a hyperbola if $ac - b^2 < 0$. Equations of different types have quite different properties. We will not go into any depth on this subject, but we will discuss some interesting properties of elliptic equations later.

The methods that we introduce here can be used to solve these and other equations in any number of dimensions. However we will focus on the problem of solving a PDE in which there are two variables. We will begin by consider the problem of solving the heat equation, for $n = 1$, subject to some additional conditions.

Example 6.1. We wish to solve the one dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \quad 0 \le x \le 1, t > 0, \tag{6.4}$$

subject to the boundary conditions $u(0,t) = u(1,t) = 0$ and the initial condition $u(x,0) = f(x)$, for some nonzero function f. Here k is a nonzero constant.

Solution. We have seen that this equation can be solved by Laplace transform. We could try this here, but instead we use a different method. A very important method of solution to this problem is to look for a separable solution. What this means is that we let

$$u(x,t) = X(x)T(t),$$

for some functions X and T. This is an extension of the method of separation of variables which works well for some first order ODEs.

Since $u(0,t) = X(0)T(t) = 0$ for all T, then $X(0) = 0$, because otherwise we would have $T(t) = 0$ for all T and this would give $u = 0$, which does not satisfy the given initial condition. By the same argument applied to the other boundary condition, $X(1) = 0$.

Next we note that $u_{xx} = X''(x)T(t)$ and $u_t = X(x)T'(t)$. Since u is a solution of the heat equation, we get

$$X''(x)T(t) = \frac{1}{k}X(x)T'(t) \implies \frac{X''(x)}{X(x)} = \frac{1}{k}\frac{T'(t)}{T(t)}.$$

Here we have a function of x only equal to a function of t alone, for all values of x and t. As we have noted before, this is only possible if both functions are equal to some constant which we denote λ.[1] This is called the *separation constant*. Actually it will turn out that there are infinitely many separation constants which are known as eigenvalues.

We have reduced the heat equation to the pair of ODES

$$\frac{X''(x)}{X(x)} = \frac{1}{k}\frac{T'(t)}{T(t)} = \lambda.$$

This gives us the differential equation $X''(x) = \lambda X(x)$. Clearly we would like a nonzero solution to this equation that satisfies the given boundary conditions $X(0) = X(1) = 0$. We consider three cases. The first is $\lambda > 0$. In this case the general solution of the ODE for X is

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}.$$

Applying the boundary conditions gives

$$A + B = 0 \tag{6.5}$$

$$Ae^{\sqrt{\lambda}} + Be^{-\sqrt{\lambda}} = 0. \tag{6.6}$$

Solving these simultaneous equations gives $A = B = 0$. So if $\lambda > 0$ we do not get nonzero solutions for X.

The case $\lambda = 0$ leads to $X''(x) = 0$ which has solution

$$X(x) = Ax + B.$$

Since $X(0) = B = 0$ and $X(1) = A = 0$, we do not get nonzero solutions for this case either.

This leaves the case $\lambda < 0$. Let $\lambda = -\omega^2$. The equation

$$X''(x) = -\omega^2 X(x)$$

has solution

$$X(x) = A\sin(\omega x) + B\cos(\omega x).$$

Taking $X(0) = B = 0$ we are left with $X(x) = A\sin(\omega x)$. This leaves us to satisfy the condition $X(1) = A\sin(\omega) = 0$. We can do this in two ways. We could take $A = 0$, but this gives $X = 0$ again. Fortunately, we can also insist that $\omega = n\pi$ for $n \in \mathbb{N}$, since $\sin(n\pi) = 0$. We have thus obtained the nonzero solution

$$X(x) = A\sin(n\pi x).$$

[1]To see why, notice that if $f(x) = g(t)$ for all x, t, and a is in the domain of g, then $f(x) = g(a)$ for all x. Since $g(a)$ is a constant, then $f(x)$ is a constant. Also $g(t) = f(b)$ for b in the domain of f. Hence g is constant too.

For T we get $T' = k\lambda T = -kn^2\pi^2 T$. This first order ODE has solution $T(t) = Be^{-kn^2\pi^2 t}$ where B is a constant of integration. This leads to a solution of the heat equation

$$u(x,t) = C\sin(n\pi x)e^{-kn^2\pi^2 t}$$

where C is a constant. This solution satisfies the given boundary conditions, but not the given initial condition.

How then do we satisfy the initial condition? The solution to the problem of satisfying an initial condition using separable solutions, goes back to d'Alembert, an 18th century French mathematician. He used this technique to obtain a solution of the wave equation. The idea is to take a linear combination of solutions, choosing the constants multiplying each solution in a clever way. The first person to actually determine the right way of choosing the coefficients of each solution correctly was Euler, but it was Joseph Fourier who was the first person to make extensive use of the technique.

Because the heat equation is linear we can take a superposition of solutions. That is, we can add two solutions together to obtain a third solution. Suppose that we add infinitely many solutions together. Let $\{C_n\}$ be a sequence of constants and define

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)e^{-kn^2\pi^2 t}. \tag{6.7}$$

The question of when this series actually converges and what it converges to when it does converge at all, is one that will be discussed later. Suppose that it does converge. It is a sum of solutions of the heat equation, so is itself a solution of the heat equation. Now set $t = 0$. If $u(x,0) = f(x)$ then we need the equation

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x). \tag{6.8}$$

to be satisfied.

The key to obtaining the values for C_n is the fact that

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2} & n = m. \end{cases}$$

To see how this allows us to obtain the value of C_n we write our expression as

$$f(x) = C_1\sin(\pi x) + C_2\sin(2\pi x) + C_3\sin(3\pi x) + \cdots + C_k\sin(k\pi x) + \cdots$$

Next we multiply both sides by $\sin(k\pi x)$ and integrate from 0 to 1. Then

$$\begin{aligned}\int_0^1 f(x)\sin(k\pi x)dx &= \int_0^1 C_1\sin(\pi x)\sin(k\pi x)dx\\ &+\int_0^1 C_2\sin(2\pi x)\sin(k\pi x)dx\\ &+\cdots+\int_0^1 C_k\sin^2(k\pi x)dx+\cdots\\ &= C_k\int_0^1 \sin^2(k\pi x)dx=\frac{1}{2}C_k,\end{aligned}$$

since all the other integrals on the right are zero. This immediately allows us to deduce the formula

$$C_k = 2\int_0^1 f(x)\sin(k\pi x)dx. \tag{6.9}$$

The numbers $\{C_k\}_{k=1}^{\infty}$ are known as the *Fourier sine coefficients of f*. This gives us a formal solution to our initial-boundary value problem for the heat equation.

$$u(x,t) = 2\sum_{n=1}^{\infty}\left\{\int_0^1 f(y)\sin(n\pi y)dy\right\}\sin(n\pi x)e^{-kn^2\pi^2 t}. \tag{6.10}$$

At this point all we have done is established a formal solution to the heat equation. We do not know if it is correct, because we do not know that the series of sines converges to f. To establish that it does, we must investigate such series in more detail.

6.2. **Fourier Series.** We will consider the problem of whether or not it is possible to expand a function f on the interval $(-\pi,\pi)$ as a series of sines and cosines. We begin with the observation that

$$\int_{-\pi}^{\pi}\cos(nx)dx=0,\ \int_{-\pi}^{\pi}\sin(nx)dx=0,$$

$$\int_{-\pi}^{\pi}\sin(nx)\cos(mx)dx=0, \int_{-\pi}^{\pi}\cos(nx)\cos(mx)dx=0$$

$$\int_{-\pi}^{\pi}\sin(nx)\sin(mx)dx=0,$$

$$\int_{-\pi}^{\pi}\sin^2(nx)dx=\int_{-\pi}^{\pi}\cos^2(nx)dx=\pi.$$

for n,m integers with $n\neq m$ Finally $\int_{-\pi}^{\pi}dx=2\pi$.

Now if we can write

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx)),$$

for $x \in (-\pi, \pi)$, then it follows that $f(x+2\pi) = f(x)$ for all x, since sine and cosine are both periodic with period 2π. Multiplying both sides by $\cos(nx)$ and integrating from $-\pi$ to π, we arrive at

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,\ n = 1, 2, 3... \tag{6.11}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx. \tag{6.12}$$

Similarly we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx. \tag{6.13}$$

These numbers are known as the Fourier sine and cosine coefficients. We will write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$$

to indicate that the Fourier series for the function on the right is given by the expression on the left. We are not yet claiming that a function is actually equal to its Fourier series.

Example 6.2. Let us find the Fourier expansion for the function $f(x) = x,\ -\pi < x < \pi,\ f(x+2\pi) = f(x)$ for all x. We have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = 2\frac{(-1)^{n+1}}{n}. \tag{6.14}$$

Integration also shows that $a_0 = 0$ and $a_n = 0$ for all n. Thus

$$f(x) \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx). \tag{6.15}$$

As mentioned above, we are not claiming that the function is actually equal to its Fourier series. This remains to be seen. A simple test is to plot the function against the series. Since we cannot add an infinite number of terms, we compare f with the Nth partial sum of the series. That is, we compare f with

$$S_N(x) = a_0 + \sum_{n=1}^{N}(a_n\cos(nx) + b_n\sin(nx)). \tag{6.16}$$

We will take $N = 100$. Does the Fourier series actually converge to f for this example? The graphical evidence suggests that it does, but this is not proof. We see that the sum of the first 100 terms does give a good approximation to f away from the end points of the interval. The jagged behaviour near the end points is a well known property of Fourier sine series, which is called the Gibb's phenomenon.

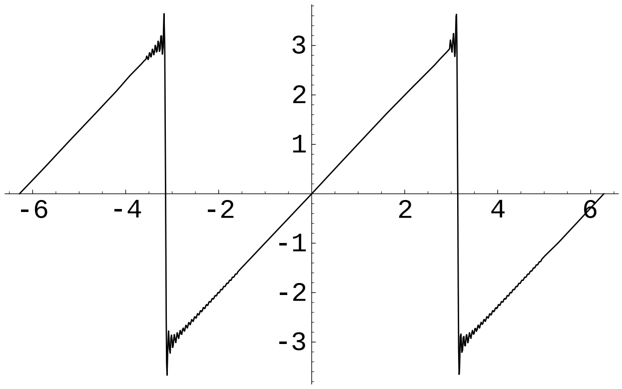

FIGURE 6. The sum of the first one hundred terms of the Fourier series for x.

Example 6.3. Now we compute the Fourier series for

$$f(x) = x^2, \ -\pi < x < \pi,$$

$f(x) = f(x + 2\pi)$ for all x. We have

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin(nx) dx = 0 \tag{6.17}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx = 4\frac{(-1)^n}{n^2}, \ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}. \tag{6.18}$$

The Fourier series is therefore

$$f(x) \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx). \tag{6.19}$$

Let us now plot the sum of the first 100 terms of this series. We see from the graph (Figure 7 below) that it gives a very good approximation to f. Notice also that the jagged behaviour near the end points that appeared with the Fourier sine series is absent for this series involving only cosines.

Example 6.4. Let us consider a more complicated example. We let $g(x) = x^3 + 3x^2 - 25\sin^2 x, \ -\pi < x < \pi, \ f(x + 2\pi) = f(x)$ for all x. Calculating the Fourier coefficients we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) dx = \frac{1}{2} \left(-25 + 2\pi^2\right). \tag{6.20}$$

For the cosine coefficients, there is a subtle feature which needs to be incorporated. For $n = 2$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(x) \cos(2x) dx = -\frac{1}{2}, \tag{6.21}$$

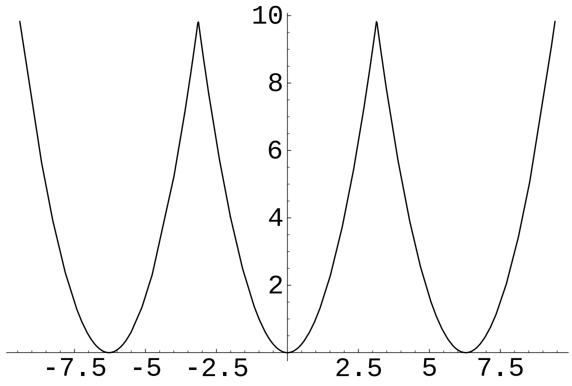

FIGURE 7. The sum of the first one hundred terms of the Fourier series for x^2.

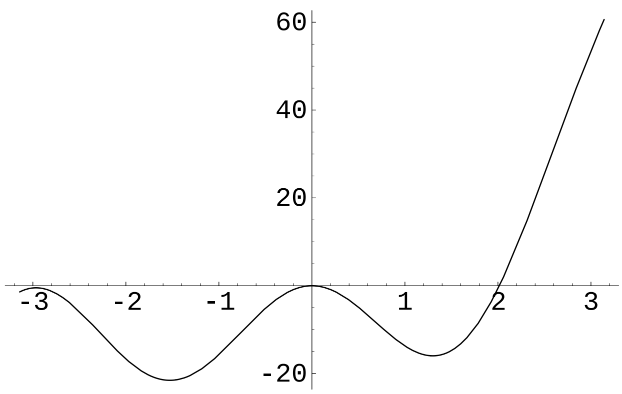

FIGURE 8. The graph of g.

but for all other non zero values of n we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2(x)\cos(2x)dx = 0. \tag{6.22}$$

For the coefficients we therefore have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)dx = 12\frac{(-1)^n}{n^2}, n \neq 2 \tag{6.23}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)dx = \frac{2(-1)^{n+1}\left(n^2\pi^2 - 6\right)}{n^3}. \tag{6.24}$$

Thus, incorporating the extra cosine term at $n = 2$, we have

$$g(x) \sim \frac{1}{2}(2\pi^2 - 25) + \frac{25}{2}\cos(2x) + \sum_{n=1}^{\infty}\left(12\frac{(-1)^n}{n^2}\cos(nx) + \frac{2(-1)^{n+1}\left(n^2\pi^2 - 6\right)}{n^3}\sin(nx)\right).$$

From the graph the match appears to be quite good.

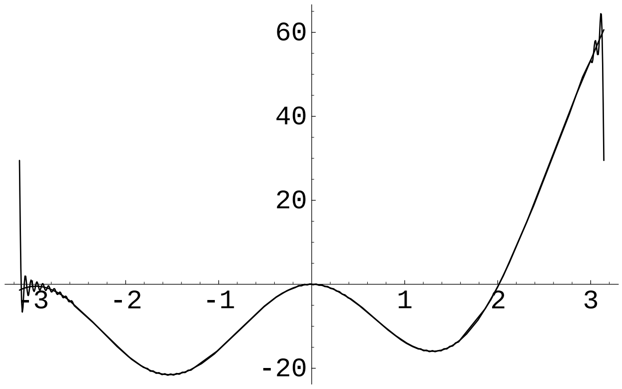

FIGURE 9. The graph of g against the sum of the first 100 terms of the Fourier series.

6.2.1. *Even and Odd Functions.* The evaluation of Fourier series is aided by the observation that if f is an odd function, then

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right) \\ &= \frac{1}{2\pi}\left(-\int_{0}^{\pi} f(x)dx + \int_{0}^{\pi} f(x)dx\right) = 0. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{0} f(x)\cos(nx)dx + \int_{0}^{\pi} f(x)\cos(nx)dx\right) \\ &= \frac{1}{2\pi}\left(-\int_{0}^{\pi} f(x)\cos(nx)dx + \int_{0}^{\pi} f(x)\cos(nx)dx\right) = 0. \end{aligned}$$

We have used $f(-x) = -f(x)$ and used the change of variables $x \to -x$ in the first integrals. Similarly if f is even, we easily show that $b_n = 0$. Thus the Fourier series for even functions contain only cosines and the Fourier series for odd functions contain only sines.

Another important result about Fourier coefficients is the Riemann-Lebesgue Lemma.

Lemma 6.2 (Riemann-Lebesgue). *Let f be differentiable on $[-\pi, \pi]$. Then*

$$\lim_{n\to\infty}\int_{0}^{\pi} f(x)\sin(nx)dx = \lim_{n\to\infty}\int_{-\pi}^{0} f(x)\sin(nx)dx = 0 \qquad (6.25)$$

$$\lim_{n\to\infty}\int_{0}^{\pi} f(x)\cos(nx)dx = \lim_{n\to\infty}\int_{-\pi}^{0} f(x)\cos(nx)dx = 0 \qquad (6.26)$$

The proof is a tutorial exercise.

6.3. **The Convergence of Fourier Series.** The question of whether a Fourier series converges and to what it converges, is in general a difficult one to answer. There are still interesting problems in the subject, but a good deal is known. We can actually make very strong statements if we make moderate assumptions on the function f. Let us first consider an example where we can prove convergence easily.

Example 6.5. Consider the Fourier series for $f(x) = x^2$ on $(-\pi, \pi)$ given by

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx). \tag{6.27}$$

Now $f_n(x) = 4\frac{(-1)^n}{n^2} \cos(nx)$ satisfies $|f_n(x)| \le \frac{4}{n^2}$ and $\sum_{n=1}^{\infty} \frac{4}{n^2} < \infty$. So the Fourier series converges uniformly. Since each function is continuous, the limit is a continuous function.

We have established that a particular Fourier series converges to a continuous function, but this does not tell us that the limit function is the original function. The graphical data suggests that the Fourier series does recover the original function, but this is not proof.

We might anticipate that the convergence of a Fourier series is related to the smoothness of the function, because the rate at which the Fourier coefficients decay is determined by the differentiability of the function.

Definition 6.3. We say that a function f is of order $g(x)$, written $O(g(x))$ if there is a constant C such that for x large $|f(x)| \le C|g(x)|$.

Proposition 6.4. *Let f be twice continuously differentiable and 2π periodic. Then the Fourier coefficients a_n, b_n are both $O(1/n^2)$.*

Proof. Consider the sine coefficients.

$$\begin{aligned}
\pi b_n &= \int_{-\pi}^{\pi} f(x) \sin(nx) dx \\
&= \left[-f(x)\frac{\cos(nx)}{n}\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} f'(x) \cos(nx) dx \\
&= \frac{1}{n^2}\left[f'(x)\sin(nx)\right]_{-\pi}^{\pi} - \frac{1}{n^2}\int_{-\pi}^{\pi} f''(x)\sin(nx)dx \\
&= -\frac{1}{n^2}\int_{-\pi}^{\pi} f''(x)\sin(nx)dx.
\end{aligned}$$

We used the fact that f is periodic, so $f(\pi) = f(-\pi)$ and cos is even. So that

$$|b_n| \le \frac{1}{\pi n^2}\int_{-\pi}^{\pi} |f''(x)|dx. \tag{6.28}$$

We take $C = \pi^{-1}\int_{-\pi}^{\pi} |f''(x)|dx$. The proof for a_n is basically identical. □

We can relax the boundary conditions on f but we will not consider this. Generally speaking, the more derivatives of f that exist, the faster the coefficients a_n, b_n will decay.

Differentiability cannot be relaxed. It is possible to find examples of continuous functions, but non-differentiable functions, whose Fourier series do not converge at all. However, if we assume differentiability, then we can establish convergence. The first theorem of this type was proved by Dirichlet.

Theorem 6.5 (Dirichlet)**.** *Let f be a piecewise differentiable function on $(-\pi, \pi)$ and suppose that its Fourier series is*

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)),$$

where the Fourier coefficients are as stated above. If f is also continuous at x_0, then the Fourier series converges to $f(x_0)$ at x_0. If f is piecewise continuous at x_0, then the Fourier series converges to $\frac{1}{2}[f(x_0^+) + f(x_0^-)]$ at x_0. Here

$$f(x_0^+) = \lim_{x \to x_0+} f(x), \; f(x_0^-) = \lim_{x \to x_0-} f(x).$$

Proof. Let

$$S_n(x_0) = a_0 + \sum_{k=1}^{n}(a_n \cos(kx_0) + b_n \sin(kx_0)).$$

We wish to show that $\lim_{n\to\infty} S_n(x_0) = \frac{1}{2}[f(x_0^+) + f(x_0^-)]$. To this end we observe that replacing the Fourier coefficients with their integral expressions gives us

$$\begin{aligned} S_n(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(t) + 2\sum_{k=1}^{n} f(t)[\cos(kt)\cos(kx_0) + \sin(kt)\sin(kx_0)]\right) dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos(k(t-x_0))\right] dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) D_n(t-x_0) dt. \end{aligned}$$

Here $D_n(\theta)$ is the so called Dirichlet kernel[2] defined by

$$D_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n}\cos(k\theta). \tag{6.29}$$

[2] If we define $(Tf)(y) = \int_a^b f(x)k(x,y)dx$, the function $k(x,y)$ is known as the kernel of the operator T.

Now if g has period 2π then $\int_{-\pi-x}^{\pi-x} g(t)dt = \int_{-\pi}^{\pi} g(t)dt$ for any x. Thus putting $u = t - x_0$ we have

$$\begin{aligned} S_n(x_0) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+u)D_n(u)du \\ &= \frac{1}{\pi}\int_{-\pi}^{0} f(x_0+u)D_n(u)du + \frac{1}{\pi}\int_{0}^{\pi} f(x_0+u)D_n(u)du. \end{aligned}$$

Now we need to know more about D_n. We can actually sum the expression for the kernel explicitly. Since $\cos(k\theta) = \Re(e^{ik\theta})$ we have

$$\sum_{k=1}^{n}\cos(k\theta) = \Re\left(\sum_{k=1}^{n} e^{ik\theta}\right) = \Re\left(\frac{e^{i\theta}\left(e^{in\theta}-1\right)}{e^{i\theta}-1}\right).$$

Here $\Re$ denotes the real part. The right side is a geometric sum with common ratio $e^{i\theta}$, which is easily summed. If $\cos(k\theta) = 1$, the sum is n. Now

$$\begin{aligned} \frac{e^{i\theta}\left(e^{in\theta}-1\right)}{e^{i\theta}-1} &= \frac{e^{i\theta}\left(e^{in\theta}-1\right)}{e^{i\theta/2}(e^{i\theta/2}-e^{-i\theta/2})} \\ &= \frac{e^{i\theta/2}\left(e^{in\theta}-1\right)}{2i\sin(\theta/2)} \\ &= \frac{e^{i(n+1/2)\theta}\left(e^{in\theta/2}-e^{-in\theta/2}\right)}{2i\sin(\theta/2)} \\ &= \frac{e^{i(n+1/2)\theta}\sin(n\theta/2)}{\sin(\theta/2)}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{2} + \Re\left(\sum_{k=1}^{n} e^{ik\theta}\right) &= \frac{1}{2} + \cos([n+\frac{1}{2}]\theta)\frac{\sin(n\theta/2)}{\sin(\theta/2)} \\ &= -\sin^2\left(\frac{n\theta}{2}\right) + \cot\left(\frac{\theta}{2}\right)\sin\left(\frac{n\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right) + \frac{1}{2} \\ &= \frac{1 - 2\sin^2\left(\frac{n\theta}{2}\right) + 2\cot\left(\frac{\theta}{2}\right)\sin\left(\frac{n\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)}{2} \\ &= \frac{\cos^2\left(\frac{n\theta}{2}\right) - \sin^2\left(\frac{n\theta}{2}\right) + 2\cot\left(\frac{\theta}{2}\right)\sin\left(\frac{n\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)}{2} \\ &= \frac{\cos(n\theta) + \cot(\theta/2)\sin(n\theta)}{2} \\ &= \frac{\sin(\theta/2)\cos(n\theta) + \cos(\theta/2)\sin(n\theta)}{2\sin(\theta/2)}. \end{aligned}$$

So we have

$$D_n(\theta) = \begin{cases} n + \frac{1}{2}, & \theta = 2N\pi, N = 0, \pm 1, \pm 2, ... \\ \dfrac{\sin([n+\frac{1}{2}]\theta)}{2\sin(\theta/2)}. & \text{otherwise.} \end{cases} \tag{6.30}$$

It is also easy to see that $\int_0^{\pi} D_n(\theta)d\theta = \int_{-\pi}^{0} D_n(\theta)d\theta = \pi/2$. In view of these integrals we can write

$$\frac{1}{2}[f(x_0^+) + f(x_0^-)] = \frac{1}{\pi}\int_0^{\pi} f(x_0^+)D_n(u)du + \frac{1}{\pi}\int_{-\pi}^{0} f(x_0^-)D_n(u)du.$$

Notice that $f(x_0^+), f(x_0^-)$ do not depend on u and so can be taken outside the integrals, and the integrals cancel half the factor of $1/\pi$. Because of all this, we have

$$S_n(x_0) - \frac{1}{2}[f(x_0^+) + f(x_0^-)] = \frac{1}{\pi}(A_n(x_0) + B_n(x_0)),$$

where

$$A_n(x_0) = \int_0^{\pi} [f(x_0 + u) - f(x_0^+)]D_n(u)du, \tag{6.31}$$

$$B_n(x_0) = \int_{-\pi}^{0} [f(x_0 + u) - f(x_0^-)]D_n(u)du. \tag{6.32}$$

If we can show that $A_n(x_0), B_n(x_0) \to 0$ as $n \to \infty$, then we are done. We will only consider the case for $A_n(x_0)$ since the case for $B_n(x_0)$ is basically the same. Using the expression for D_n we can write

$$A_n(x_0) = \int_0^{\pi} \frac{f(x_0 + u) - f(x_0^+)}{u}\frac{u/2}{\sin(u/2)}\sin([n + \frac{1}{2}]u)du. \tag{6.33}$$

Using the expansion for $\sin(A + B)$ this can be expressed as

$$A_n(x_0) = \int_0^{\pi} \phi(u)\cos(nu)du + \int_0^{\pi} \psi(u)\sin(nu)du$$

where

$$\phi(u) = \frac{f(x_0 + u) - f(x_0^+))}{u}\frac{u}{2}, \tag{6.34}$$

$$\psi(u) = \frac{f(x_0 + u) - f(x_0^+))}{u}\frac{u/2}{\sin(u/2)}\cos(u/2). \tag{6.35}$$

We assumed that f was piecewise differentiable, so that both ϕ and ψ are piecewise continuous on $[0, \pi]$ and hence the integrals exist. So we can apply the Riemann-Lebesgue Lemma to the two integrals to conclude that $\lim_{n\to\infty} A_n(x_0) = 0$. The same argument works for $B_n(x_0)$ and the theorem follows. □

Example 6.6. Consider the function $f(x) = x$, $-\pi < x < \pi$. We computed the Fourier series for this previously. The function is differentiable and continuous for all $x \in (-\pi, \pi)$, so that at $x = \frac{\pi}{2}$, we may write

$$\pi = 2 \times 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(\frac{n\pi}{2})$$
$$= 4(1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots).$$

This is a famous series for π that was discovered in India about 1000 years ago. It is also known as Leibnitz's series.

Example 6.7. We consider the piecewise differentiable function

$$f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ x & -\pi < x < 0. \end{cases} \tag{6.36}$$

Without computing the Fourier series, we can see that at $x = 0$ the Fourier series will converge to $1/2(0+1) = 1/2$. We compute the Fourier coefficients to give

$$a_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{0} x dx + \int_{0}^{\pi} dx\right) = \frac{2-\pi}{4},$$

$$a_n = \frac{1}{2\pi}\left(\int_{-\pi}^{0} x \cos(nx) dx + \int_{0}^{\pi} \cos(nx) dx\right)$$
$$= -\frac{-1 + (-1)^n}{n^2\pi},$$

and

$$b_n = \frac{1}{2\pi}\left(\int_{-\pi}^{0} x \sin(nx) dx + \int_{0}^{\pi} \sin(nx) dx\right)$$
$$= -\frac{-1 + (-1)^n + (-1)^n\pi}{n\pi}.$$

Thus by Dirichlet's Theorem for $x \in (-\pi, \pi), x \neq 0$

$$f(x) = \frac{2-\pi}{4} - \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n^2\pi} \cos(nx)$$
$$+ \sum_{n=1}^{\infty} \frac{-1 + (-1)^n + (-1)^n\pi}{n\pi} \sin(nx)).$$

The graph shows that at $x = 0$, the Fourier series passes through 0.5.

It is quite important to realise that the assumption in Dirichlet's Theorem that f is differentiable, at least piecewise, cannot be relaxed. That is, it is not sufficient to assume only that f is continuous. It is possible to find continuous functions whose Fourier series diverge at infinitely many points.

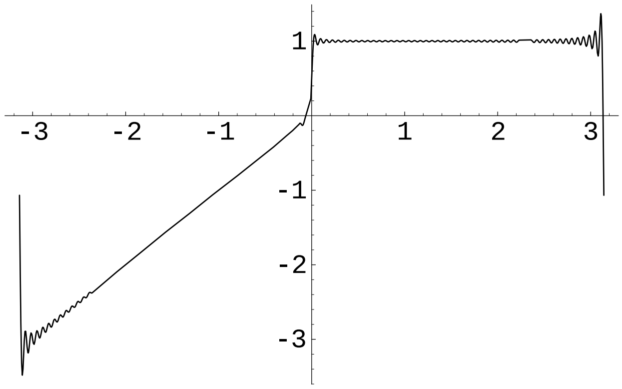

FIGURE 10. The graph of f against the sum of the first 100 terms of the Fourier series.

6.3.1. *The Gibbs-Wilbraham Phenomenon.* In Figure 6.7 we can see an example of what is commonly called the Gibbs Phenomenon. This is the jagged behaviour of the graph of the partial sums as we approach a point of discontinuity. Henry Wilbraham was actually the first person to observe this phenomenon in a paper in 1848, which was largely ignored. In 1898 Albert Michelson observed the phenomenon arising in a machine he had constructed and in 1899 Gibbs wrote a paper which gave an explanation of the phenomenon. Maxime Bocher provided a detailed analysis in 1906 and called it the Gibbs phenomenon. It is often now called the Gibbs-Wilbraham Phenomenon, in recognition of Wilbraham's earlier work.

The phenomenon is illustrated clearly by Figure 6.7. It is the successive undershooting and overshooting of the limit at the discontinuity x_0 as the number of terms in the partial sums $S_N(f, x_0)$ increase. It also refers to the fact that this overshooting and undershooting does not die out as N increases, but actually converges to a limit.

The point where the maximum value of the overshooting (undershooting) occurs moves closer and closer to the point of discontinuity, but the height of the overshoot (undershoot) converges to a fixed value. The reason for this is that Dirichlet's Theorem tells us that at the jump discontinuity the Fourier series converges to $1/2(f(x_0^+) + f(x_0^-))$. The graph of the Fourier series approximation near a discontinuity cannot be smooth because the actual function is not smooth there. The height of the overshooting (undershooting) can be shown to converge to $0.089a$ where $a = f(x_0^+) - f(x_0^-)$. That is, the maximum overshoot (undershoot) is about 9% of the jump at the discontinuity.

6.4. **Parseval's Theorem.** A very useful fact about Fourier series is that they preserves square integrals in a very precises sense.

Theorem 6.6. *Let f be a piecewise differentiable function on $(-\pi, \pi)$. Suppose that the Fourier sine coefficients are b_n and the cosine coefficients are a_n. Then the following equality holds*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2). \tag{6.37}$$

Sketch of proof. Let us briefly consider why this result is true. We will not consider the question of convergence. Let us suppose that f is differentiable, so that we know from Dirichlet's Theorem that

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)). \tag{6.38}$$

So that

$$\int_{-\pi}^{\pi} f(x)^2 dx = \int_{-\pi}^{\pi}\left(a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))\right)^2 dx \tag{6.39}$$

Now $\int_{-\pi}^{\pi} \cos^2(nx)dx = \int_{-\pi}^{\pi} \sin^2(nx)dx = \pi$, $\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = 0$ etc. When we expand the right side out we get terms of the form

$$\int_{-\pi}^{\pi} a_n^2 \cos^2(nx)dx, \int_{-\pi}^{\pi} b_n^2 \sin^2(nx)dx,$$

$$\int_{-\pi}^{\pi} a_n a_m \cos(nx)\cos(mx)dx, \int_{-\pi}^{\pi} b_n b_m \sin(nx)\sin(mx)dx,$$

$$\int_{-\pi}^{\pi} a_n b_m \cos(nx)\sin(mx)dx, \int_{-\pi}^{\pi} b_n b_m \sin(nx)\cos(mx)dx,$$

$$\int_{-\pi}^{\pi} a_0 a_m \cos(nx)dx, \int_{-\pi}^{\pi} a_0 b_n \sin(nx)$$

and the first term is $\int_{-\pi}^{\pi} a_0^2 dx = 2\pi a_0^2$. Now $\int_{-\pi}^{\pi} a_n^2 \cos^2(nx)dx = a_n^2\pi$, $\int_{-\pi}^{\pi} b_n^2 \sin^2(nx)dx = b_n^2\pi$, $\int_{-\pi}^{\pi} a_n a_m \cos(nx)\cos(mx)dx = 0$ etc. Combining these gives the result. The case when f is piecewise differentiable is a little more involved, since we need to be careful about what happens on the left hand side at any point where there is a jump discontinuity in f. However it can be shown that the resulting integral does indeed equal $\int_{-\pi}^{\pi}(f(x))^2 dx$. □

Example 6.8. We have computed the Fourier series for $f(x) = x$ previously. We found that $a_n = 0$ and $b_n = 2\frac{(-1)^n}{n}$. So by Parseval's equality

$$\sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty}\frac{4}{n^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3}. \tag{6.40}$$

If we rearrange this we have the famous result of Euler that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{6.41}$$

Example 6.9. Now let us consider the case when $f(x) = x^2$. The Fourier coefficients are $b_n = 0$, $a_n = 4\frac{(-1)^n}{n^2}$, $a_0 = \frac{\pi^2}{3}$. From Parseval's Theorem we find that

$$2\frac{\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{2\pi^4}{5}. \tag{6.42}$$

From this we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{6.43}$$

This is another formula originally due to Euler.

Example 6.10. As a final example, we consider the Fourier series of $f(x) = x^3$, $-\pi < x < \pi$, $f(x+2\pi) = f(x)$. Then the cosine coefficients are all zero and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin(nx) dx = \frac{2(-1)^{n+1}\left(n^2\pi^2 - 6\right)}{n^3}. \tag{6.44}$$

Now

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^3 dx = \frac{2\pi^6}{7},$$

hence

$$\sum_{n=1}^{\infty} b_n^2 = \frac{2\pi^6}{7}. \tag{6.45}$$

Now

$$b_n^2 = \frac{4\pi^4}{n^2} - \frac{48\pi^2}{n^4} + \frac{144}{n^6}.$$

Consequently

$$\sum_{n=1}^{\infty} \left(\frac{4\pi^4}{n^2} - \frac{48\pi^2}{n^4} + \frac{144}{n^6}\right) = \frac{2\pi^6}{7}. \tag{6.46}$$

So we can rearrange this to get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{144}{n^6} &= \frac{2\pi^6}{7} + \sum_{n=1}^{\infty} \frac{48\pi^2}{n^4} - \sum_{n=1}^{\infty} \frac{4\pi^4}{n^2} \\
&= \frac{2\pi^6}{7} + 48\pi^2\frac{\pi^4}{90} - 4\pi^6\frac{\pi^2}{6} \\
&= \frac{16\pi^6}{105}.
\end{aligned}$$

From which we deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}. \tag{6.47}$$

Again, this is a result of Euler.

Actually Euler was able to produce a general formula for $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$. However to this day, there is no known method of evaluating the corresponding sums when the exponent is odd. So for example, we do not know the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^3}$. Apery proved in 1978 that the sum is irrational and the sum is now known as Apery's constant. It is also known that one of

$$\sum_{n=1}^{\infty} \frac{1}{n^5}, \sum_{n=1}^{\infty} \frac{1}{n^7}, \sum_{n=1}^{\infty} \frac{1}{n^9}, \sum_{n=1}^{\infty} \frac{1}{n^{11}}$$

must be irrational, but not which one. This is an example of a curious phenomenon in mathematics. Some problems in the case where a constant or dimension is even are quite solvable, whereas the corresponding problems in the odd case appear to be impossible.

The function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{6.48}$$

is known as the Riemann Zeta function. Euler showed the remarkable relation of the function to the prime numbers.

Theorem 6.7 (Euler)**.** *Let s be a complex number. Then*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^s} &= \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \frac{1}{\left(1-\frac{1}{2^s}\right)} \frac{1}{\left(1-\frac{1}{3^s}\right)} \frac{1}{\left(1-\frac{1}{5^s}\right)} \frac{1}{\left(1-\frac{1}{7^s}\right)} \frac{1}{\left(1-\frac{1}{11^s}\right)} \cdots\end{aligned}$$

The product is taken over all the prime numbers. Because of the relation, the ζ function plays a central role in the theory of prime numbers. It is called the Riemann zeta function because in one of the most famous papers in mathematics, Riemann showed how the function may be used to establish deep properties of the prime numbers and essentially invented the branch of mathematics known as analytic number theory. If s is allowed to be complex it turns out that $\zeta(s) = 0$ has solutions. Riemann conjectured that every complex zero has the form $s = 1/2 + it$ for $t \in \mathbb{R}$. This is known as the Riemann Hypothesis and it is widely considered to be the most important unsolved problem in mathematics.

6.4.1. *Fourier Series on* $(-L, L)$. Suppose that we wish to expand a function which is defined on the interval $(-L, L)$ and has period $2L$. This can easily be obtained by a change of variables. Suppose that f has period $T = 2L$. We introduce the function $g(x) = f\left(\frac{T}{2\pi}x\right)$. Then it is clear that

$$g(x + 2\pi) = f\left(\frac{T}{2\pi}x + T\right) = f\left(\frac{T}{2\pi}x\right) = g(x).$$

Thus g has period 2π and so we can expand it in a standard Fourier series. We therefore write

$$f\left(\frac{T}{2\pi}x\right) \sim a_0 + \sum \left(a_n \cos(nx) + b_n \sin(nx)\right), \tag{6.49}$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right) dx, \tag{6.50}$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right) \cos(nx)dx, \tag{6.51}$$

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right) \sin(nx)dx. \tag{6.52}$$

But if we put $t = Tx/2\pi$, then, using $T = 2L$ the above becomes

$$f(t) \sim a_0 + \sum \left(a_n \cos(\frac{n\pi t}{L}) + b_n \sin(\frac{n\pi t}{L})\right), \tag{6.53}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt, \tag{6.54}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi t}{L})dt, \tag{6.55}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi t}{L})dt. \tag{6.56}$$

We are thus able to expand a function as a Fourier series on any symmetric interval. Dirichlet's Theorem is unaltered, and Parseval's Theorem becomes

$$\frac{1}{L}\int_{-L}^{L} |f(t)|^2 dt = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2). \tag{6.57}$$

Example 6.11. Let $f(t) = t^2, -1 < t < 1$ $f(t+2) = f(t)$ for all t. The function is even, so the sine terms are all zero. We also have

$$a_0 = \frac{1}{2}\int_{-1}^{1} t^2 dt = \frac{1}{3},\ a_n = \int_{-1}^{1} t^2 \cos(n\pi t)dt = \frac{4(-1)^n}{n^2\pi^2}. \tag{6.58}$$

Thus for $t \in (-1, 1)$

$$t^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2}\cos(n\pi t). \tag{6.59}$$

6.5. **Half Range Expansions.** To solve the heat equation, we needed to expand a function f on the interval $[0, 1]$ in terms of sines. Is this possible? To see how to achieve this, we introduce the notion of even and odd extensions. Let f be defined on the interval $[0, L)$. We wish to extend it to the interval $(-L, L)$. We can do this in any number of ways. But there are two quite simple ones. First, we introduce the even extension. We define

$$f_E(x) = \begin{cases} f(x) & x \in [0, L) \\ f(-x) & x \in (-L, 0). \end{cases}$$

This is an even function which matches f on $(0, L)$. Because it is an even function, we can expand it in a series of cosines on the interval $(-L, L)$. Simple algebra shows that the Fourier cosine coefficients may be calculated as

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \tag{6.60}$$

and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \tag{6.61}$$

If f is piecewise differentiable, then the Fourier series for f_E will converge to $f(x)$ on $[0, L$ and to $f(-x)$ on $(-L, 0)$.

Similarly we can introduce the odd extension. We define

$$f_E(x) = \begin{cases} f(x) & x \in [0, L) \\ -f(-x) & x \in (-L, 0). \end{cases}$$

We also must have $f(0) = 0$. This is an odd function which matches f on $(0, L)$. Because it is an odd function, we can expand it in a series of sines on the interval $(-L, L)$. The sine coefficients are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{6.62}$$

The sine series will converge to f on $[0, L)$ and to $-f(-x)$ on $(-L, 0)$.

The point is that we can expand a piecewise differentiable function on $[0, L)$ as either a series of sines or cosines. The cosine series will converge to the even extension and the sine series to the odd extension.

What happens at L? This depends on how the function behaves at zero. The examples should make this clear.

Example 6.12. Let us expand $f(x) = x$ as a sine series on $[0, 1)$. The coefficients are

$$b_n = 2\int_0^1 x \sin(n\pi x)dx = 2\frac{(-1)^{n+1}}{n\pi}. \tag{6.63}$$

We plot the sum of the first 100 terms on the whole of $(-1, 1)$ below. Notice that at $L = \pm 1$ we see the Gibbs phenomenon appearing. This

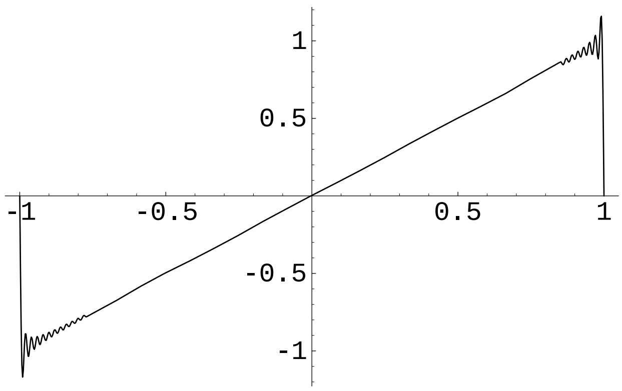

FIGURE 11. The graph of f against the sum of the first 100 terms of the Fourier sine series.

is because the periodised function is not continuous. If we now compute the cosine series we find that $a_0 = 1/2$ and

$$a_n = \frac{2\,(-1 + (-1)^n)}{n^2\pi^2}.$$

Now let us plot the sum of the first 100 terms of the cosine series.

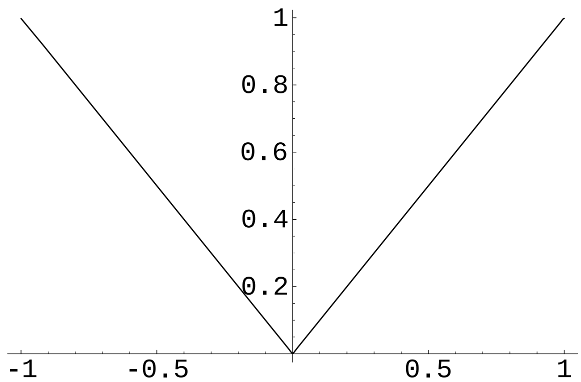

FIGURE 12. The graph of f against the sum of the first 100 terms of the Fourier cosine series.

Here the periodised function is continuous, so there is no Gibbs phenomenon.

Observe that the Fourier sine series returns an odd function, the cosine series an even function. They both return f on the interval $[0, 1)$.

6.5.1. *Complex Fourier Series.* Since $e^{ix} = \cos x + i \sin x$, it is possible to rewrite the theory of Fourier series in terms of complex exponentials. The theory is not very much changed, but the formula that we end up with tend to be more compact. Given f we define the nth Fourier coefficient of f as

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} d\varphi.$$

Some authors prefer the equivalent form

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{0}^{2\pi} f(\varphi) e^{-in\varphi} d\varphi.$$

As long as the range of integration is 2π, then there is no practical difference.

We can recover f from the Fourier coefficients by writing

$$f(\theta) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{in\theta}. \tag{6.64}$$

General treatments of Fourier series often use the complex form because it is technically more convenient, though it is equivalent to the form we have already seen. As before, the Fourier coefficients decay to zero as n increases.

Lemma 6.8 (Riemann-Lebesgue)**.** *Let f be continuous on I and periodic on $\mathbb{R}$ with period 2π. Then* $\lim_{n\to\infty} |\widehat{f}(n)| = 0$.

Proof. We have

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} d\varphi \\
&= \frac{1}{2\pi} \int_{-\pi+\frac{\pi}{n}}^{\pi+\frac{\pi}{n}} f(\varphi + \frac{\pi}{n}) e^{-in(\varphi+\frac{\pi}{n})} d\varphi \\
&= -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi + \frac{\pi}{n}) e^{-in\varphi} d\varphi.
\end{aligned}$$

So that

$$2\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(\varphi) - f(\varphi + \frac{\pi}{n})) e^{-in\varphi} d\varphi.$$

Hence

$$|\widehat{f}(n)| \leq \frac{1}{4\pi} \int_{-\pi}^{\pi} |f(\varphi) - f(\varphi + \frac{\pi}{n})| d\varphi.$$

The integrand is bounded, so we can take the limit as $n \to \infty$ through the integral. Thus

$$\begin{aligned} 0 \le \lim_{n\to\infty} |\widehat{f}(n)| &\le \lim_{n\to\infty} \frac{1}{4\pi} \int_{-\pi}^{\pi} |f(\varphi) - f(\varphi + \frac{\pi}{n})| d\varphi \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \lim_{n\to\infty} |f(\varphi) - f(\varphi + \frac{\pi}{n})| d\varphi = 0. \end{aligned}$$

□

Convergence of the Fourier series is covered by the complex form of Dirichlet's Theorem, which we present in the case where f is differentiable. The proof is similar to that of the previous version of the theorem.

Theorem 6.9 (Dirichlet)**.** *Suppose that $f \in L^1(I)$ and that $f'(\theta_0)$ exists. ($I = [-\pi, \pi]$, or equivalently $[0, 2\pi)$.) Let*

$$(S_N f)(\theta) = \sum_{n=-N}^{N} \widehat{f}(n) e^{in\theta}.$$

Then $\lim_{N\to\infty}(S_N f)(\theta_0) = f(\theta_0)$. That is, the Fourier series for f converges to f at a point where f is differentiable.

Fourier series for

$$f \in L^2(I) = \left\{ f : I \to \mathbb{C} : \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \right\},$$

are very well behaved. The most important result is due to Riesz and Fischer. We will not prove this result. We can of course replace $[-\pi, \pi]$ with $[0, 2\pi]$.

Theorem 6.10 (Riesz-Fischer)**.** *Suppose that $f \in L^2(I)$. Then*

$$\lim_{N\to\infty} \|S_N f - f\|_2 = 0,$$

where $\|h\|_2 = \left(\int_{-\pi}^{\pi} |h(\theta)|^2 d\theta\right)^{1/2}$. Further, if we define

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx,$$

then

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 = \|f\|^2. \tag{6.65}$$

Conversely, suppose that $\{a_n\}_{n=-\infty}^{\infty}$ is a two sided sequence of complex numbers such that $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$. Then there is a unique $f \in L^2(I)$ such that $a_n = \widehat{f}(n)$ for each n.

6.6. **More on Separation of Variables.** We have seen that the heat equation can be solved by means of separation of variables. Using Dirichlet's Theorem we see that

$$\begin{aligned} u_t &= k u_{xx}, \;\; 0 < x < 1, t > 0 \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= f(x) \end{aligned}$$

has a solution given by

$$u(x,t) = 2\sum_{n=1}^{\infty} \left\{ \int_0^1 f(y)\sin(n\pi y)dy \right\} \sin(n\pi x) e^{-n^2\pi^2 t}. \tag{6.66}$$

whenever f is piecewise differentiable.

Example 6.13. We take $f(x) = x(1-x)$ and set $k = 1$. Then

$$b_n = 2\int_0^1 y(1-y)\sin(n\pi y)dy = 4\frac{1-(-1)^n}{n^3\pi^3}. \tag{6.67}$$

So the solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} 4\frac{1-(-1)^n}{n^3\pi^3}\sin(n\pi x)e^{-n^2\pi^2 t} \\ &= \sum_{n=1}^{\infty} \frac{8}{(1+2n)^3\pi^3}\sin((2n+1)\pi x)e^{-(2n+1)^2\pi^2 t}. \end{aligned} \tag{6.68}$$

Let us see what the solution looks like. Notice that it decays very quickly as t increases. This is because of the exponential factors $e^{-n^2\pi^2 t}$ in the solution.

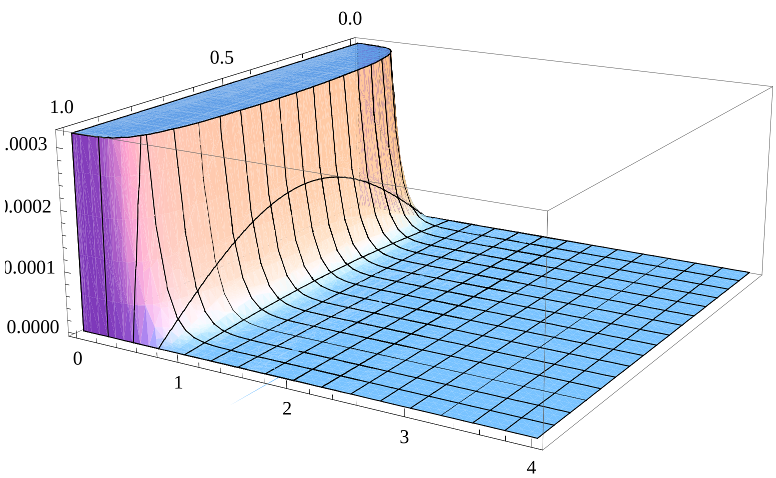

FIGURE 13. The solution for $0 \le x \le 1, \; 0 \le t \le 2$.

We have so far considered only the simplest problem for the heat equation. There are many more kinds of problems which can be solved by Fourier series methods. For example, suppose we want to solve

$$u_t = u_{xx}, \; x \in [0, L], \; t \geq 0, \tag{6.69}$$

$$u(x, 0) = f(x), u_x(0, t) = u_x(L, t) = 0. \tag{6.70}$$

Again we use separation of variables, but now the equation $X'' = \lambda X$ has the boundary conditions $X'(0) = X'(L) = 0$ and the eigenfunctions are of the form $\cos\left(\frac{n\pi x}{L}\right)$. The solution is then given by a Fourier cosine series. We leave this as an exercise.

We might have a more general problem like

$$u_t = u_{xx}, \; x \in [0, L], \; t \geq 0, \tag{6.71}$$

$$u(x, 0) = f(x), u(0, t) = q(t), u(L, t) = p(t), \tag{6.72}$$

where q and p are given functions of t. The approach we take here is to look for a solution of the form

$$u(x, t) = v(x, t) + \frac{L - x}{L} q(t) + \frac{x}{L} p(t),$$

where $v(0, t) = v(L, t) = 0$ and $v(x, 0) = f(x) - \frac{L-x}{L} q(0) - \frac{x}{L} p(0)$. Observe also that $u_{xx} = v_{xx}$ and

$$u_t = v_t + \frac{L - x}{L} q'(t) + \frac{x}{L} p'(t).$$

So that v must satisfy

$$v_t = v_{xx} + F(x, t) \tag{6.73}$$

$$v(x, 0) = h(x), v(0, t) = v(L, t) = 0 \tag{6.74}$$

where $h(x) = f(x) - \frac{L-x}{L} q(0) - \frac{x}{L} p(0)$ and $F(x, t) = -\frac{L-x}{L} q'(t) - \frac{x}{L} p'(t)$. This leads us to consider how we can solve an equation with a source term like $F(x, t)$. Physically we can think of F as the heat being put into the system by an external source.

To solve this problem we look for a solution of the form

$$v(x, t) = \sum_{n=1}^{\infty} c_n(t) \sin(\lambda_n x) e^{-\lambda_n^2 t} \tag{6.75}$$

where the eigenvalues are $\lambda_n = n\pi/L$. We further assume that we can write F as a Fourier series

$$F(x, t) = \sum_{n=1}^{\infty} F_n(t) \sin(\lambda_n x) e^{-\lambda_n^2 t} \tag{6.76}$$

with

$$F_n(t) e^{-\lambda_n^2 t} = \frac{2}{L} \int_0^L F(x, t) \sin(\lambda_n x) dx. \tag{6.77}$$

Substituting our expressions for v and F into the PDE we get $c_n'(t) = F_n(t)$ with $c_n(0) = \frac{2}{L}\int_0^L h(x)\sin(\lambda_n x)dx = a_n$. So that

$$c_n(t) = a_n + \int_0^t F_n(s)ds.$$

This leads to the solution

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x)e^{-\lambda_n^2 t} + \sum_{n=1}^{\infty}\left(\int_0^t F_n(s)ds\right)\sin(\lambda_n x)e^{-\lambda_n^2 t}. \tag{6.78}$$

We can solve other boundary value problems for the heat equation by variations of this method. Now however we apply our Fourier series methods to the wave equation.

6.7. **The Wave Equation.** We solve the following problem for the wave equation in one space dimension.

$$\begin{aligned} \frac{1}{c^2}u_{tt} &= u_{xx} \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x). \end{aligned}$$

We assume that f, g are both piecewise differentiable.

The wave equation models the behavior of a wave propagating in the x direction as t varies. For example, consider a guitar string fixed at two points. The initial profile of the string will be a straight line above the neck and sound board of the instrument. When plucked, the string will vibrate, producing sound. The height of the string at at any point along its length at time t, will be given by $u(x,t)$, where u satisfies the wave equation. In the problem stated here, we also specify the initial velocity of the wave, $u_t(x,0)$. The end points of the string are fixed at height 0.

Once more we look for a separable solution. This means setting $u(x,t) = X(x)T(t)$. Then, as with the heat equation, we have $X(0) = X(L) = 0$ and further

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = \lambda, \tag{6.79}$$

where λ is the separation constant.

The problem for X is exactly the same as for the heat equation. There are three possible cases, with $\lambda = k^2 > 0, \lambda = 0, \lambda = -k^2$. Choosing $\lambda = k^2$ gives $X(x) = Ae^{kx}+Be^{-kx}$ and the conditions $X(0) = X(L) = 0$ lead to conclusion that $A = B = 0$. Thus we cannot get a nonzero X from this choice of λ.

Similarly the case $\lambda = 0$ also leads to $X(x) = 0$, just as for the heat equation case. Finally we let $\lambda = -k^2$ and this gives

$$X(x) = A\cos(kx) + B\sin(kx).$$

The boundary conditions $X(0) = X(L) = 0$ give $A = 0$ and $kL = n\pi$. Hence we can conclude that

$$X(x) = B\sin\left(\frac{n\pi}{L}x\right),$$

and $k = \frac{n\pi}{L}$.

Thus $\lambda = -\frac{n^2\pi^2}{L^2}$, which means that

$$T'' = -\frac{n^2\pi^2c^2}{L^2}T.$$

Solving gives

$$T(t) = C\cos\left(\frac{n\pi c}{L}t\right) + D\sin\left(\frac{n\pi c}{L}t\right) \tag{6.80}$$

and the separable solutions of the wave equation satisfying our boundary conditions are

$$u(x,t) = B\sin\left(\frac{n\pi}{L}x\right)\left(C\cos\left(\frac{n\pi c}{L}t\right) + D\sin\left(\frac{n\pi c}{L}t\right)\right). \tag{6.81}$$

The idea then is to satisfy the initial conditions by taking a linear combination of solutions to give

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}t\right) + B_n\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi c}{L}t\right)\right]. \tag{6.82}$$

Since $u(x,0) = f(x)$ we require

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi}{L}x\right). \tag{6.83}$$

As in the heat equation case

$$A_n = \frac{2}{L}\int_0^L f(y)\sin\left(\frac{n\pi}{L}y\right)dy. \tag{6.84}$$

We also have $u_t(x,0) = g(x)$ and so differentiating (6.82) with respect to t, gives

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n\sin\left(\frac{n\pi}{L}x\right). \tag{6.85}$$

Therefore $\frac{n\pi c}{L}B_n$ must be equal to the Fourier sine coefficients of g, giving

$$B_n = \frac{2}{n\pi c}\int_0^L g(y)\sin\left(\frac{n\pi}{L}y\right)dy. \tag{6.86}$$

A solution of the wave equation satisfying the given conditions is therefore

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L f(y) \sin\left(\frac{n\pi}{L} y\right) dy \right) \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi c}{L} t\right) + \sum_{n=1}^{\infty} \left(\frac{2}{n\pi c} \int_0^L g(y) \sin\left(\frac{n\pi}{L} y\right) dy \right) \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi c}{L} t\right). \tag{6.87}$$

Example 6.14. We take $L = 1, f(x) = x(1-x), g(x) = x$. Then

$$A_n = 2\int_0^1 y(1-y)\sin(n\pi y)dy = 4\frac{1-(-1)^n}{n^3\pi^3}, \tag{6.88}$$

$$B_n = \frac{2}{n\pi c}\int_0^1 y\sin(n\pi y)dy = 2\frac{(-1)^{n+1}}{n^2\pi^2 c}. \tag{6.89}$$

This leads to the solution

$$u(x,t) = \sum_{n=1}^{\infty} 4\frac{1-(-1)^n}{n^3\pi^3}\sin(n\pi x)\cos(n\pi ct) + \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n^2\pi^2 c}\sin(n\pi x)\sin(n\pi ct).$$

Now we plot the solution for $0 \le t \le 5$. We take $c = 1$ for convenience.

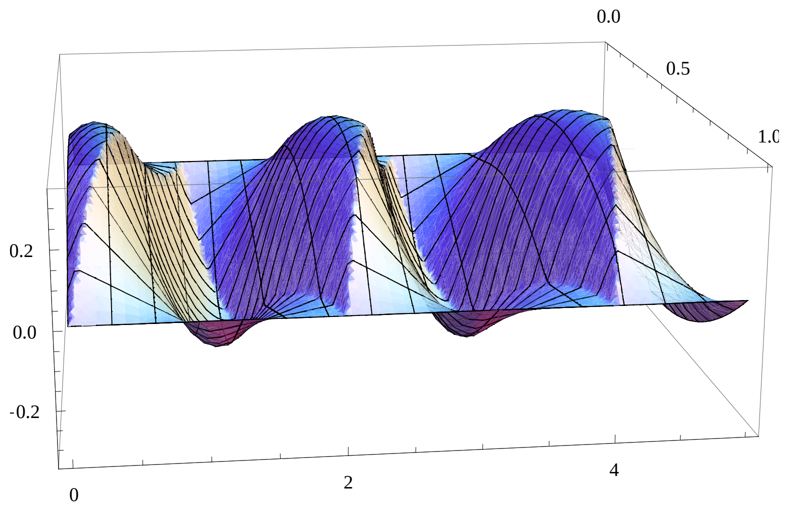

FIGURE 14. The solution of the wave equation for $c = 1,\ 0 \le x \le 1,\ 0 \le t \le 5$.

The wave like behaviour of the solution is clear from the three dimensional graph. The wave equation is one of the most important equations in physics. In three dimensions it is used to model radio and other electromagnetic waves, as well as water waves, sound waves and many other phenomena.

6.8. **Laplace's Equation.** The third equation that we will consider is the Laplace equation. This is the equation

$$\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = \Delta u = 0. \tag{6.90}$$

The operator $\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ is called the Laplacian. We will focus on Laplace's equation for $n = 2$. However some general considerations will be in order.

Equations of different types have different properties. For elliptic equations, one of the most important properties is the maximum principle. We will state it for the Laplace equation.

Theorem 6.11. *Let $\Delta u = 0$ on the domain $\Omega \subset \mathbb{R}^n$. Then the maximum and minimum values of the solution u will occur on the boundary of Ω.*

An application of this result follows.

Theorem 6.12. *Consider the Dirichlet problem for the Laplace equation: Solve*

$$\Delta u = 0, \ \ x \in \Omega \subset \mathbb{R}^n \tag{6.91}$$

$$u\big|_{\partial\Omega} = f, \tag{6.92}$$

where $\partial\Omega$ is the boundary of Ω. If a solution to this problem exists, then it is unique.

In the Dirichlet problem, we solve the Lalplace equation on some region of space, and insist that the solution be given by a known function f on the boundary of the domain.

Proof. Suppose that u, v both satisfy the Dirichlet problem. We wish to show that $u = v$. Observe that $w = u - v$ satisfies Laplace's equation. Moreover,

$$w\big|_{\partial\Omega} = u\big|_{\partial\Omega} - v\big|_{\partial\Omega} = f - f = 0.$$

Thus w is a solution of Laplace's equation which is equal to zero on the boundary of Ω. But by the maximum principle, the maximum and minimum values of w are both zero, so $w = 0$. But this means that $u = v$. □

Another important result is Harnack's inequality.

Theorem 6.13 (Harnack)**.** *If u is continuous on the closed ball $|x - x_0| \leq R$ on $\mathbb{R}^n$ centered at x_0 and harmonic on its interior, then for every point x with $|x - x_0| = r < R$,*

$$\frac{1-(r/R)}{[1+(r/R)]^{n-1}}u(x_0) \leq u(x) \leq \frac{1+(r/R)}{[1-(r/R)]^{n-1}}u(x_0).$$

On $\mathbb{R}^2$ the inequality can be written:

$$\frac{R-r}{R+r}u(x_0) \leq u(x) \leq \frac{R+r}{R-r}u(x_0).$$

This has many consequences. A simple one is that if $u(x_0) = 0$, then the solution is zero everywhere.

The problem of studying Laplace's equation is so important that it makes up its own branch of mathematics, which is known as Potential Theory. The maximum principle and Harnack's inequality are fundamental to this subject.

We will solve Laplace's equation by separation of variables, as we did the wave and heat equations.

Example 6.15. We solve Laplace's equation on a rectangular region.

$$u_{xx} + u_{yy} = 0, \; 0 \leq x \leq a, \; 0 \leq y \leq b$$
$$u(0, y) = 0, u(a, y) = 0, u(x, b) = 0, u(x, 0) = f(x).$$

As before we look for a separable solution, setting $u(x, y) = X(x)Y(y)$. This leads to

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda, \tag{6.93}$$

where λ is the separation constant. We require $X(0) = X(a) = 0$ and $Y(b) = 0$. Applying the arguments as we used in the heat and wave equation examples, we find that $\lambda = -\frac{n^2\pi^2}{a^2}$ and $X(x) = B\sin\left(\frac{n\pi}{a}x\right)$. Solving the equation for Y gives

$$Y(y) = C\cosh\left(\frac{n\pi}{a}y\right) + D\sinh\left(\frac{n\pi}{a}y\right). \tag{6.94}$$

The condition $Y(b) = 0$ gives

$$C = -D\frac{\sinh\left(\frac{n\pi}{a}y\right)}{\cosh\left(\frac{n\pi}{a}y\right)}. \tag{6.95}$$

This gives

$$Y(y) = -D\text{sech}\left(\frac{bn\pi}{a}\right)\sinh\left(\frac{n\pi(b-y)}{a}\right). \tag{6.96}$$

We therefore have

$$u(x, y) = -BD\sin\left(\frac{n\pi}{a}x\right)\frac{\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)}. \tag{6.97}$$

Seeking a solution by using the superposition of solutions, we find that

$$u(x,y)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{a}x\right)\frac{\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)}. \tag{6.98}$$

so that

$$u(x,0)=f(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{a}x\right)\tanh\left(\frac{n\pi b}{a}\right). \tag{6.99}$$

So we have

$$A_n=\frac{2}{a\tanh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx. \tag{6.100}$$

The unique solution of Laplace's equation with these conditions is therefore

$$u(x,y)=\sum_{n=1}^{\infty}\left(\frac{2\int_0^a f(z)\sin\left(\frac{n\pi}{a}z\right)dz}{a\tanh\left(\frac{n\pi b}{a}\right)}\right)\sin\left(\frac{n\pi}{a}x\right)\frac{\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)}.$$

Let us evaluate this for $a=b=1$, $f(x)=x(1-x)$. The Fourier coefficients were computed above for the wave equation example. The graph of the solution is given below. Observe that both the maximum

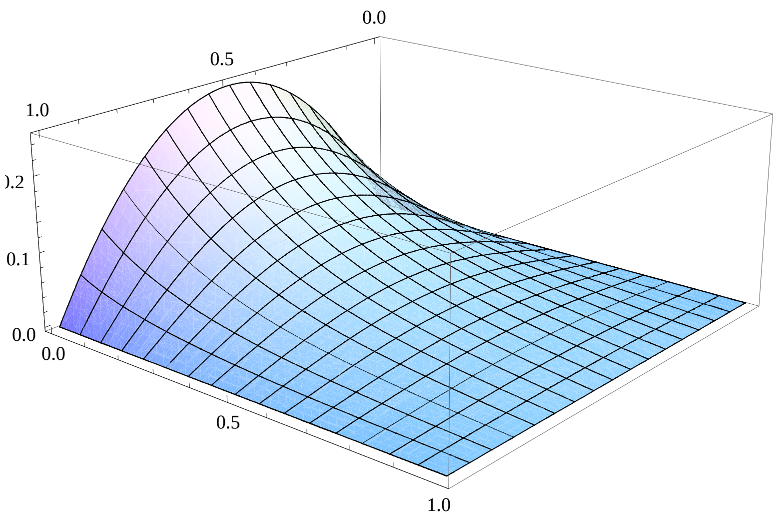

FIGURE 15. The solution of the Laplace equation.

and minimum values of the solution occur on the boundary of the square.

6.9. **The Poisson Integral Formula for the Disc.** In many problems, we are required to solve Laplace's equation on a circular region, with the value of the solution specified on the boundary. A famous theorem in complex analysis, called the Riemann Mapping Theorem, tells us that any simply connected region in the complex plane can be transformed into the unit disc, by a *conformal mapping* – a transformation which preserves angles between vectors.

Laplace's equation arises in many physical applications. For example, consider the wing of an aircraft. A major problem in aircraft design is to calculate how much lift there will be on a wing at a given height and velocity. For this calculation we have to solve Laplace's equation. But a cross section of a wing is an odd shape, and solving Laplace's equation on this region is difficult. Therefore, a conformal mapping is applied and the whole problem is converted to solving Laplace's equation on the disc. Then the solution is mapped back to give the solution on the original region and so we can work out how much lift our aircraft will have.

Consequently, it is of great importance to be able to solve Laplace's equation on a disc, subject to a given boundary condition. We only discuss the simplest case of the problem.

We solve the Laplace equation on the disc

$$D_R = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 < R^2\},$$

subject to the conditions $u(R, \theta) = f(\theta)$ and $u(r, \theta)$ is finite as $r \to 0^+$.
Note: This is clearly the *Dirichlet problem* for the Laplace equation on a disc.

We convert Laplace's equation to polar coordinates. That is, we let $x = r\cos\theta$ and $y = r\sin\theta$. By using the chain rule, we see that in polar coordinates the Laplace equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{6.101}$$

We look for a separable solution $u(r, \theta) = V(r)\Phi(\theta)$. Then the Laplace equation is

$$(V''(r) + \frac{1}{r}V'(r))\Phi(\theta) + \frac{1}{r^2}V\Phi''(\theta) = 0, \tag{6.102}$$

which separates to give

$$\frac{1}{V}(r^2V''(r) + rV'(r)) = -\frac{1}{\Phi}\Phi''(\theta). \tag{6.103}$$

Since the left hand side is a function of r only and the right hand side is a function of θ only, then both the left and right must equal some constant λ. Thus

$$\frac{1}{V}(r^2V''(r) + rV'(r)) = \lambda. \tag{6.104}$$

We will see that it is convenient to set $\lambda = n^2$, where n is an integer. Consequently

$$r^2 V''(r) + rV'(r) - n^2 V(r) = 0. \tag{6.105}$$

This is an Euler type equation and it has solutions of the form $V(r) = r^a$. Substitution shows that a must satisfy the quadratic equation $a^2 - n^2 = 0$. This gives the general solution for V as

$$V(r) = Ar^n + Br^{-n} \tag{6.106}$$

where A and B are constants.

Now we insist that u is finite as $r \to 0^+$. Since our solution is $u = V\Phi$, this tells us that $B = 0$, since $r^{-n} \to \infty$ as $r \to 0^+$.

The equation for Φ is $\Phi''(\theta) = -n^2\Phi(\theta)$ which has solution $\Phi(\theta) = Ce^{in\theta} + De^{-in\theta}$. Hence the solution of the Laplace equation that we obtain from separation of variables with this choice of λ is

$$u(r, \theta) = Ar^n(Ce^{in\theta} + De^{-in\theta}),\ n = 0, 1, 2, 3, ...$$

We want to satisfy the boundary condition $u(a, \theta) = f(\theta)$. To this end we try a superposition of solutions. We set

$$\begin{aligned} u(r, \theta) &= \sum_{n=0}^{\infty} r^n (A_n e^{in\theta} + A_{-n} e^{-in\theta}) \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} A_n e^{in\theta}. \end{aligned} \tag{6.107}$$

If $u(R, \theta) = f(\theta),\ 0 \le \theta < 2\pi$ then we obtain

$$u(R, \theta) = \sum_{n=-\infty}^{\infty} R^{|n|} A_n e^{in\theta} = f(\theta). \tag{6.108}$$

This is a Fourier series for f. (Which is the reason why $\lambda = n^2$ is the 'right' choice to make). The Fourier coefficients A_n are given by

$$A_n = \frac{1}{2\pi R^{|n|}} \int_0^{2\pi} f(\varphi) e^{-in\varphi} d\varphi. \tag{6.109}$$

This gives the solution

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (\frac{r}{R})^{|n|} \left[\int_0^{2\pi} f(\varphi) e^{in(\theta - \varphi)} d\varphi \right] \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) \sum_{n=-\infty}^{\infty} (\frac{r}{R})^{|n|} e^{in(\theta-\varphi)} d\varphi. \end{aligned} \tag{6.110}$$

We are permitted to swap integral and sum because the series converges uniformly in the disc D_R. We may easily sum the geometric series to obtain

$$\sum_{n=-\infty}^{\infty} (\frac{r}{R})^{|n|} e^{in(\theta-\varphi)} = 1 + \sum_{n=1}^{\infty} (\frac{r}{R})^n e^{in(\theta-\varphi)} + \sum_{n=1}^{\infty} (\frac{r}{R})^n e^{-in(\theta-\varphi)}$$
$$= 1 + \frac{\frac{r}{R} e^{i(\theta-\varphi)}}{1 - \frac{r}{R} e^{i(\theta-\varphi)}} + \frac{\frac{r}{R} e^{-i(\theta-\varphi)}}{1 - \frac{r}{R} e^{-i(\theta-\varphi)}}$$
$$= \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\varphi) + r^2}.$$

We used Euler's formula $e^{iz} = \cos z + i \sin z$ here.

Hence the Dirichlet problem for the disc D_R has solution

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\varphi)(R^2 - r^2)}{R^2 - 2rR\cos(\theta-\varphi) + r^2} d\varphi. \tag{6.111}$$

The function

$$K(r, R, \theta, \varphi) = \frac{(R^2 - r^2)}{R^2 - 2rR\cos(\theta-\varphi) + r^2},$$

is known as the Poisson kernel for the disc D_R. It is an example of a Green's function. A great deal more may be said on the subject of Green's functions, but that is beyond the scope of this course.

7. Numerical methods for Differential Equations

7.1. Numerical Methods for ODES based on Taylor Series.

7.1.1. *Euler's Method.* Euler's method for numerically solving ODEs is the simplest of all methods. Recall that if y is a function of a single variable, which is differentiable in a region containing the point a, then for h small enough, we can write

$$y'(a) \approx \frac{y(a+h) - y(a)}{h}$$

Hence

$$y(a+h) \approx y(a) + hy'(a).$$

Now, since $y'(x) = f(x, y(x))$ we immediately get the linear approximation

$$y(a+h) \approx y_0 + hf(a, y_0) = y_1, \tag{7.1}$$

in which $y_0 = y(a)$. Having obtained an approximate value for $y(a+h)$ from (7.1), we can now approximate the value of y at $a + 2h$. This would be

$$y(a+2h) \approx y_1 + hf(a+h, y_1) = y_2. \tag{7.2}$$

We can repeat this process as many times as we like. This leads to a simple algorithm for solving a differential equation numerically.

We choose a natural number $n > 0$. Let $h = (b-a)/n$. We call h the step size. Then we set $x_i = x_0 + ih$, where $x_0 = a$. Now we let $y_i = y(x_i)$. Combining this with the relation (7.1) we obtain Euler's method

Algorithm 1 Euler's Method.

To solve the IVP $y' = f(x, y(x)), y(a) = y_0$ let the step size $h = (b-a)/n$. Set $x_i = x_0 + ih$. Let $y_i = y(x_i)$. To obtain an approximation to the solution of the given IVP, at the points x_i generate iterates according to

$$y_{i+1} = y_i + hf(x_i, y_i). \tag{7.3}$$

End of Algorithm

(1) Euler's method is easy to code because of its simplicity.

(2) Euler's method has the drawback that in order to obtain a good approximation to the solution, it is usually necessary to take h small.

(3) The method is based upon approximating the solution by the first order Taylor polynomial about the initial point $x = a$.

Example 7.1. Approximate the solution of $y'(x) = y + 3x$ on the interval $0 \le x \le 1$ with y satisfying the initial condition $y(0) = 1$. Take step sizes equal to $h = \frac{1}{10}$.

First we solve the equation exactly so that we can compare our numerical solution with the true solution. We observe that the equation is first order linear, hence we need to obtain an integrating factor. This is $e^{\int -1dx} = e^{-x}$. Thus the equation can be written

$$e^{-x}y'(x) - e^{-x}y = \frac{d}{dx}\left(e^{-x}y(x)\right) = 3xe^{-x}.$$

Integrating and applying the initial condition leads to the exact solution

$$y(x) = 4e^x - 3(1+x).$$

Now we have to solve the equation numerically, using Euler's method. Here, the equation is $y' = y + 3x$. So $f(x, y) = y + 3x$. We also have $x_0 = 0$ and $y(0) = y_0 = 1$. Hence

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + \frac{1}{10}(y_0 + 3x_0) = 1 + \frac{1}{10}(1 + 3 \times 0) = 1.1$$

This is the approximation for the value of the solution at the point $x_0 + h = 0.1$. We compare to the true value and see that $y(0.1) = 1.12068$. So the first approximation is in error by 0.02. Next we compute y_2. Recall that $x_1 = x_0 + h$. So

$$y_2 = y_1 + hf(y_1, x_1) = 1.1 + \frac{1}{10}(1.1 + 3 \times 0.1) = 1.24$$

Whereas the true value is $y(0.2) = 1.28561$. Continuing the process we obtain the following results

x_i	Euler y_i	Exact value of $y(x_i)$	Absolute error
0	1	1	0
0.1	1.1	1.12068	0.02068
0.2	1.24	1.28561	0.04561
0.3	1.424	1.49944	0.07544
0.4	1.6564	1.7673	0.1109
0.5	1.94204	2.09489	0.15285
0.6	2.28624	2.48848	0.20224
0.7	2.69487	2.95501	0.26014
0.8	3.17436	3.50216	0.3278
0.9	3.73179	4.13841	0.40662
1.0	4.37497	4.87313	0.49816

The column marked Euler y_i contains the values of the numerical solution obtained by Euler's method. The table also contains the true values for comparison, and the absolute error of the approximation in the final column.

Notice that the error gets worse the more iterations we take. That is, the further we get from our starting point $x_0 = 0$, the more inaccurate our numerical solution becomes.

What is happening is that the errors from the previous steps are accumulating. As noted above, to obtain good approximations with Euler method, we usually have to make h very small. But this makes the method slow and inefficient. If we had to obtain a numerical solution on a large interval, say for example $[0, 1000]$, in order to have any hope of obtaining an accurate solution across the whole interval, we would have to take h so small that the method would be totally impractical.

Euler's method is simple, but it is not very good. Fortunately, there are better methods available.

7.2. **Taylor series methods.** One obvious way of improving the accuracy of the Euler method is to extend it to include higher order derivatives of y. Recall that Taylor's theorem states that if y is $n+1$ times differentiable in an interval I around $x = a$, then there exists a number $\xi \in I$ such that

$$\begin{aligned} y(a+h) = y(a) + hy'(a) + \frac{1}{2}h^2 y''(a) + \cdots + \frac{1}{n!}h^n y^{(n)}(a) \\ + \frac{1}{(n+1)!}h^{n+1}y^{(n+1)}(\xi). \end{aligned} \tag{7.4}$$

We used Taylor series methods to find series solutions of linear differential equations earlier in the notes. Taylor series methods can be quite naturally extended to solve nonlinear equations as well, and they are especially useful for solving differential equations numerically.

Actually, it is easy to see that the Euler method is a Taylor series method in which we use the first two terms of the Taylor polynomial (7.4). This suggests that the natural extension of the Euler method is to take higher order terms in the Taylor series (7.4).

This presents the problem of how to compute the higher order derivatives $y^{(n)}(a)$. In fact, we can do this by the chain rule. Observe that $y'(x) = f(x, y(x))$. Applying the chain rule gives

$$\begin{aligned} y''(x) &= \frac{d}{dx}\left(y'(x)\right) = \frac{d}{dx}f(x, y(x)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \\ &= f_x(x, y(x)) + f(x, y(x))f_y(x, y(x)). \end{aligned} \tag{7.5}$$

The third derivative can be computed in the same manner.

$$y'''(x) = \frac{d}{dx}\left(y''(x)\right) = \frac{d}{dx}\left(f_x(x, y(x)) + f(x, y(x))f_y(x, y(x))\right).$$

It should be obvious that we can compute derivatives of y to all orders by the same means. It is just a question of how many derivatives we want. (And how much patience we have!) Computing more derivatives, and using them to approximate y should give a more accurate numerical solution to the IVP.

This leads us to the *kth order Taylor series method for the IVP of Definition 1.1*

Algorithm 2 kth order Taylor series method.

To solve the ODE $y' = f(x, y(x)), y(a) = y_0$ let the step size $h = (b-a)/n$. Set $x_i = x_0 + ih$. Let $y_i = y(x_i)$. Define

$$T_k(x, y; h) = f(x, y) + f'(x, y)\frac{h}{2} + \cdots + f^{(k-1)}(x, y)\frac{h^{k-1}}{k!}. \tag{7.6}$$

To obtain an approximation to the solution of the given IVP, at the points x_i generate iterates according to

$$y_{i+1} = y_i + hT_k(x_i, y_i; h). \tag{7.7}$$

End of Algorithm

Example 7.2. Use a second order Taylor series method to approximate the solution of $y'(x) = y+3x$ on the interval $0 \le x \le 1$ with y satisfying the initial condition $y(0) = 1$. Take step sizes equal to $h = \frac{1}{10}$.

We first have to compute the second derivative of y. Since $y'(x) = y(x) + 3x$, we see that

$$y''(x) = \frac{d}{dx}y'(x) = \frac{d}{dx}(y(x) + 3x) = y'(x) + 3 = 3 + 3x + y(x).$$

We thus have the iterative scheme

$$y_{i+1} = y_i + h(f(x_i, y_i) + \frac{h^2}{2}\frac{d}{dx}(f(x, y(x)))\Big|_{x=x_i} \tag{7.8}$$

$$= y_i + h(y_i + 3x_i) + \frac{h^2}{2}(3 + 3x_i + y_i)$$

Recalling that $h = 0.1, x_0 = 0$ and starting with $y_0 = 1$ we get

$$y_1 = 1 + 0.1(1 + 3 \times 0) + \frac{(0.1)^2}{2}(3 + 3 \times 0 + 1) = 1.12.$$

Next

$$y_2 = 1.12 + 0.1(1.12 + 3 \times 0.1) + \frac{(0.1)^2}{2}(3 + 3 \times 0.1 + 1.12) = 1.2841.$$

The following table contains the complete list of results.

x_i	Taylor y_i	Exact value of $y(x_i)$	Absolute error
0	1	1	0
0.1	1.12	1.12068	0.0068
0.2	1.2841	1.28561	0.00151
0.3	1.49693	1.49944	0.00251
0.4	1.76361	1.7673	0.00369
0.5	2.08979	2.09489	0.0051
0.6	2.48171	2.48848	0.00677

x_i	Taylor y_i	Exact value of $y(x_i)$	Absolute error
0.7	2.94629	2.95501	0.00872
0.8	3.49116	3.50216	0.011
0.9	4.12473	4.13841	0.01368
1.0	4.85632	4.87313	0.01681

The column marked Taylor y_i contains the numerical solution given by the second order Taylor scheme. It is obvious that the second order Taylor series method is considerably more accurate than the simple Euler method. However, notice that the accuracy decreases the more steps we take. We could of course obtain higher accuracy by computing higher and higher order derivatives. But because the Taylor series method involves taking what are effectively polynomial approximations for $y(a + h)$, the method will always give results which *eventually* diverge from the true solution. In other words, the further from the initial condition we are, the less accurate our numerical approximation will be.

Taylor series methods also have the drawback that they require evaluation of higher derivatives of y. This can be computationally expensive. For this reason, we should seek other, more efficient methods of obtaining numerical solutions for our IVP.

7.3. **Runge-Kutta methods.** One of the drawbacks of the Taylor series method is the need to evaluate derivatives of $f(x, y(x))$. One way to avoid this problem is to use an approach due to the German mathematicians C. Runge and M.W. Kutta. The Runge-Kutta method is as follows. As before, we have a single starting value for y, namely $y_0 = y(a)$. Values of y at equally spaced points are then obtained by an iterative scheme of the form

$$y_{i+1} = y_i + h\phi(x_i, y_i, h). \tag{7.9}$$

The problem is to choose the function ϕ in such a way that it returns the same values as an nth order Taylor series method, without the need to evaluate derivatives of $f(x, y(x))$. How is this achieved? As we will see, there is in fact no unique way of doing it.

We start with a second order Runge-Kutta method. That is, with $n = 2$. For a second order Taylor series method, we have the iterative scheme

$$y_{i+1} = y_i + hT_2(x_i, y_i, h) \tag{7.10}$$

in which (neglecting the error term)

$$T_2(x_i, y_i, h) = f(x_i, y_i) + \frac{h}{2}[f_x(x_i, y_i) + f_y(x_i, y_i)f(x_i, y_i)].$$

Now we want to produce a form of ϕ which matches this expression, but does not require any differentiation. The idea is to sample the function $f(x, y)$ at different points. That is, we sample f at x_i, y_i but also at $(x_i + \alpha h, y_i + \alpha f(x_i, y_i))$ for some value of α. Given this, the iterative scheme for y_i can be written

$$y_{i+1} = y_i + h[A_1 f(x_i, y_i) + A_2 f(x_i + \alpha h, y_i + \alpha h f(x_i, y_i))]. \quad (7.11)$$

How do we choose α, A_1, A_2 in order that (7.11) matches the Taylor scheme? First, expand $f(x_i + \alpha h, y_i + \alpha h f(x_i, y_i))$ in a Taylor series. We have

$$\begin{aligned} f(x_i + \alpha h, y_i + \alpha h f(x_i, y_i)) &= f(x_i, y_i) + f_x(x_i, y_i)\alpha h \\ &\quad + f_y(x_i, y_i)\alpha h f(x_i, y_i) + R_2(x_i, y_i) \end{aligned}$$

in which R_2 is the remainder term. It is not hard to show that the remainder term has the form $R_2 = Ch^2$, for some constant C. Now we use this in our expression for ϕ. Collecting all the terms together gives

$$\phi(x_i, y_i, h) = (A_1 + A_2) f(x_i, y_i) + A_2 h[\alpha f_x(x_i, y_i) + \alpha f_y(x_i, y_i) + Ch]$$

If we compare this to the Taylor scheme and neglect the remainder term, we see that we must have

$$A_1 + A_2 = 1 \quad A_2\alpha = \frac{1}{2} \quad (7.12)$$

It is now clear that there is no unique solution for these equations. We must make some choice in order to produce an actual numerical scheme. We have seen similar situations before. Different choices lead to different schemes, each with its own advantages and disadvantages.

The second order Runge-Kutta scheme with the choice $\alpha = \frac{1}{2}$ is known as the modified Euler method. In this instance we have $A_2 = 1, A_1 = 0$. The scheme is

$$y_{i+1} = y_i + hf\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i)\right) \quad (7.13)$$

The second order Runge-Kutta scheme with $\alpha = 1$ is known as Heun's method. It is also called the improved Euler method. For this choice of α, we have $A_1 = A_2 = \frac{1}{2}$. It can be represented as

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))]. \quad (7.14)$$

Example 7.3. Use Heun's method and the modified Euler method to solve the IVP $y'(x) = 3xy(x), y(0) = 2$ on the interval $[0, 1]$, taking $h = 0.1$.

Solution First we use the modified Euler method. Our starting point is $y_0 = 2$. Here, $f(x, y) = 3xy$. The exact solution of this IVP is $y = 2e^{\frac{3}{2}x^2}$.

The modified Euler method for this example takes the form

$$
\begin{aligned}
y_{i+1} &= y_i + hf\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i)\right) \\
&= y_i + 0.1\left(3\left(x_i + \frac{0.1}{2}\right)\left(y_i + \frac{0.1}{2} 3x_i y_i\right)\right) \qquad (7.15)
\end{aligned}
$$

Working out the value of y_1 we have

$$y_1 = 2 + 0.1\,(3(0 + 0.05)(2 + 0.05 \times 3(0)(2))) = 2.03.$$

From this we get y_2 :

$$y_2 = 2.03 + 0.1\left(3(0.1 + 0.05)(2.03 + \frac{0.1}{2} 3 \times 0.1 \times 2.03)\right) = 2.12272.$$

Compiling our results and comparing with the exact values, we obtain the following table.

x_i	Mod Euler y_i	Exact value of $y(x_i)$	Absolute error
0	2	2	0
0.1	2.03	2.03023	0.00023
0.2	2.12272	2.12367	0.00094
0.3	2.2867	2.28907	0.00237
0.4	2.53761	2.5425	0.00489
0.5	2.90074	2.90998	0.00924
0.6	3.41526	3.43201	0.01675
0.7	4.14117	4.17096	0.02979
0.8	5.17077	5.22339	0.05262
0.9	6.64754	6.74059	0.09305
1.0	8.79786	8.96338	0.16582

Now we do the same using Heun's method.

$$
\begin{aligned}
y_{i+1} &= y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))] \\
&= 2 + \frac{0.1}{2}[3x_i y_i + 3(x_{i+1}(y_i + 0.1 \times 3x_i y_i))]
\end{aligned}
$$

For y_1 we get

$$y_1 = 2 + \frac{0.1}{2}\,(3 \times 0 \times y_0 + 3(0.1 \times (y_0 + 3 \times 0.1 \times 0 \times y_0)) = 2.03.$$

Continuing the process we get the following table of results.

x_i	Heun y_i	Exact value of $y(x_i)$	Absolute error
0	2	2	0
0.1	2.03	2.03023	0.000226129
0.2	2.12318	2.12367	0.000496093

x_i	Heun y_i	Exact value of $y(x_i)$	Absolute error
0.3	2.28815	2.28907	0.000925716
0.4	2.54076	2.5425	0.00173892
0.5	2.90663	2.90998	0.0033541
0.6	3.42546	3.43201	0.00655177
0.7	4.15817	4.17096	0.0127957
0.8	5.19854	5.22339	0.024851
0.9	6.6926	6.74059	0.0479852
1	8.87105	8.96338	0.092333

Notice that Heun's method is more accurate than the modified Euler method. It may at first glance seem rather odd that we get different answers for the different methods, since we have set the scheme up so that it agrees to second order with the Taylor method. Why do we get different answers?

The reason is that we have not taken into consideration the error term. Our choice for α, A_1 and A_2 will effect the error term Ch^2, since the constant C depends on these numbers. So different choices for the parameters will lead to schemes with different degrees of accuracy.

By the same analysis as above, it is possible to derive Runge-Kutta schemes for higher orders. The method is as follows. We want a numerical scheme of the form

$$y_{i+1} = y_i + \phi(x_i, y_i, h). \tag{7.16}$$

We require the function ϕ to match the kth order Taylor expansion. That is

$$\phi(x_i, y_i, h) = T_k(x_i, y_i, h) + O(h^2). \tag{7.17}$$

The function ϕ has the general form

$$\phi(x_i, y_i, h) = \sum_{j=1}^{m} A_j K_j(x_i, y_i, h).$$

Such a scheme is known as an m stage, kth order method. Now the terms K_j are computed in the following way. We always have

$$K_1(x_i, y_i, h) = f(x_i, y_i).$$

For $2 \le j \le m$, K_j is defined in terms of the weighted average of the previous terms. It satisfies

$$K_j(x_i, y_i, h) = f(x_i + \alpha_j h, y_i + h \sum_{r=1}^{j-1} \beta_{jr} K_r(x_i, y_i, h)).$$

The parameters α_j satisfy $0 \le \alpha_j \le 1$ and $\alpha_j = \sum_{r=1}^{j-1} \beta_{jr}$. To work out the values, we have to expand the functions K_j in a Taylor series, and compare to the expression T_k. This can be very tedious.

It is obvious that an m stage Runge-Kutta method requires m evaluations of f at each iteration. This can be computationally expensive, but it is usually not prohibitive.

The accuracy of Runge-Kutte methods increases with the order. However, so does the complexity of the method. It is for this reason, that the lower order methods tend to be the most commonly used.

A great deal is known about the properties of these methods. For example, Runge-Kutta methods only exist if $m \geq k$. In fact Runge-Kutta methods with $m = k$ only exist for $k = 1, 2, 3, 4$. If $k = 5$ then m will have to be equal to 6. For $k \geq 7$ we require $m \geq k + 2$.

As in the order two case, in the nth order case, there are many possible choices for the weights. Each scheme has its advantages and disadvantages and a detailed discussion of each is not practical in an introductory course like this one. Instead we will simply present some examples of the higher order schemes.

One third order Runge-Kutta scheme is of the form

$$y_{i+1} = y_i + \frac{h}{6}(K_1 + 4K_2 + K_3) \tag{7.18}$$

where

$$\begin{aligned} &K_1 = f(x_i, y_i),\ K_2 = f(x_i + \frac{h}{,} y_i + \frac{h}{2}K_1), \\ &K_3 = f(x_i + h, y_i - hK_1 + 2hK_2). \end{aligned} \tag{7.19}$$

Notice the similarity between the form of (7.18) and Simpson's rule for numerical integration. In fact if f depends on x only, then this rule is precisely Simpson's rule. This is a general observation. When f depends on x only, the problem of solving $y = f(x, y)$ is simply the problem of integrating $f(x)$. The Runge-Kutta rules in this case just reduce to the familiar numerical quadrature rules that are used to perform numerical integration, such as Simpson's rule and the Trapezoidal rule.

Fourth order schemes are particularly popular. One example is the following, which is due to Runge.

$$y_{i+1} = y_i + \frac{h}{6}[K_1 + 2K_2 + 2K_3 + K_4] \tag{7.20}$$

where

$$\begin{aligned} &K_1 = f(x_i, y_i),\ K_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}K_1), \\ &K_3 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}K_2), \\ &K_4 = f(x_i + h, y_i + hK_3). \end{aligned} \tag{7.21}$$

Notice the recursive nature of all these methods. We first have to calculate K_1 before we can calculate K_2. In turn we need K_2 in order to calculate K_3 etc. This is a feature of all Runge-Kutta schemes.

For completeness, we present another fourth order scheme due to Runge, Kutta and Gill. It is

$$y_{i+1} = y_i + \frac{h}{6}\left[K_1 + 2\left(1 - \frac{1}{\sqrt{2}}\right)K_2 + 2\left(1 + \frac{1}{\sqrt{2}}\right)K_3 + K_4\right] \tag{7.22}$$

where

$$\begin{aligned} K_1 &= f(x_i, y_i),\ K_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}K_1), \\ K_3 &= f\left(x_i + \frac{h}{2}, y_i + \left(-\frac{1}{2} + \frac{1}{\sqrt{2}}\right)hK_1 + \left(1 - \frac{1}{\sqrt{2}}\right)hK_2\right), \\ K_4 &= f\left(x_i + h, y_i - \frac{h}{\sqrt{2}}K_2 + \left(1 + \frac{1}{\sqrt{2}}\right)hK_3\right). \end{aligned} \tag{7.23}$$

This scheme is said to be the most widely used fourth order Runge-Kutta method.

The Runge-Kutta schemes of order n can be shown to have an error bound which decreases proportionally to order h^{n+1}. So for example, the error in the fourth order schemes is of the order h^5.

7.4. **Predictor-Corrector Methods.** The methods we have so far considered have all been single step methods. That is, we calculate the value of y_{i+1} from the value of y_i. In this sense, all our methods so far have been single step methods. Every single step method has the form

$$y_{i+1} = y_i + h\phi(x_i, y_i, h).$$

By contrast, for a k-step method, we use k values of the approximate solution to obtain the next value. More precisely, to solve $y' = f(x, y)$ we have an iterative scheme of the form

$$y_{n+k} = -\sum_{j=0}^{k-1} \alpha_j y_{n+j} + h\sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}). \tag{7.24}$$

These methods are also called *multi-step methods*. Notice that in the form of the k-step method (7.24), the value of y_{n+k} may be given implicitly. We will discuss this in more detail below.

First, we consider a simple example. In the Euler method, we approximated the derivative $y'(x)$ by the quotient $\frac{y(x+h)-y(x)}{h}$. We could also have used the centered difference formula:

$$y'(x) \approx \frac{y(x+h) - y(x-h)}{2h}.$$

If we have equally spaced points

$$x_n + h = x_{n+1},\ x_{n+1} + h = x_{n+2},$$

then we have

$$y'(x_{n+1}) \approx \frac{y(x_{n+2}) - y(x_n)}{2h}. \tag{7.25}$$

Then the differential equation $y' = f(x, y)$ becomes

$$\frac{y(x_{n+2}) - y(x_n)}{2h} \approx f(x_{n+1}, y(x_{n+1}) \tag{7.26}$$

Using the notation $y_i \approx y(x_i)$ we get the difference scheme

$$y_{n+2} = y_n + 2hf(x_{n+1}, y_{n+1}) \tag{7.27}$$

This is a *two step* method for solving the IVP $y' = f(x, y),\ y(a) = y_0$.

Another two step method comes from Simpson's rule. We know that

$$y(x_{n+2}) - y(x_n) = \int_{x_n}^{x_{n+2}} y'(x)dx \tag{7.28}$$

Now according to Simpson's rule

$$y(x_{n+2}) - y(x_n) \approx \frac{h}{3}\left(y'(x_n) + 4y'(x_{n+1}) + y'(x_{n+2})\right). \tag{7.29}$$

Using $y'(x) = f(x, y(x))$ we get

$$\begin{aligned} y(x_{n+2}) - y(x_n) \approx \frac{h}{3}&\left(f(x_n, y(x_n)) + 4f(x_{n+1}, y(x_{n+1}))\right) \\ &+ f(x_{n+2}, y(x_{n+2}))). \end{aligned} \tag{7.30}$$

Which leads to the two step difference scheme

$$y_{n+2} - y_n = \frac{h}{3}\left(f(x_n, y_n) + 4f(x_{n+1}, y_{n+1}) + f(x_{n+2}, y_{n+2})\right) \tag{7.31}$$

This last method (7.31) is an example of an *implicit method*. It is implicit because we do not obtain y_{n+2} directly. Rather we have to solve an equation to obtain y_{n+2}. If f is a nonlinear function of y then we will have to employ some numerical method, such as Newton's method to obtain the value of y_{n+2}. By contrast the first method (7.27) is explicit, because y_{n+2} is given directly.

In practice implicit and explicit methods are often used in pairs. This is because using implicit and explicit methods in pairs allows for the effective control of errors. Such explicit-implicit schemes are known as *predictor-corrector* pairs.

Before discussing predictor-corrector pairs in a more general setting, we make one further observation about the schemes (7.27) and (7.31). Although they both come from perfectly reasonable approximations for the integral $\int_a^b y'(x)dx$, neither is very useful for numerical work. This is because they are numerically unstable. For certain functions f these methods produce very severe errors which propagate very quickly, unless we make the step size very small.

The problem of numerical instability is one of the most important in numerical analysis. Unfortunately we do not have the time in this course to explain exactly why these apparently reasonable methods have these serious numerical difficulties associated with them. The interested student should consult a more advanced text on the numerical solution of ODES. We will now present a class of methods which have turned out to be extremely useful.

7.5. **Adams Methods as Predictor-Corrector Pairs.** Adams methods come in two forms: Explicit and implicit. The Adams methods are all based upon interpolatory quadrature formulas for an integral. For a general k step method, we take quadrature rules of the form

$$\int_{x_{n+k-1}}^{x_{n+k}} g(x)dx = h\left(A_0 g(x_n) + A_1 g(x_{n+1}) + \cdots + A_k g(x_{x_{n+k}})\right). \tag{7.32}$$

Here the numbers $A_0, A_1, ..., A_k$ are the weights. For an explicit method the last weight is always zero. That is, $A_k = 0$. Explicit Adams methods are usually called Adams-Bashforth methods and the implicit methods are known as Adams-Moulton methods. The general Adams-Bashforth method has the form

$$\begin{aligned} y_{n+k} - y_{n+k-1} = h\,(A_0 f(x_n, y_n) + A_1 f(x_{n+1}, y_{n+1}) + \cdots \\ + A_{n+k-1} f(x_{n+k-1}, y_{n+k-1}))\,. \end{aligned} \tag{7.33}$$

The general Adams-Moulton method has the form

$$\begin{aligned} y_{n+k} - y_{n+k-1} = h\,(A_0 f(x_n, y_n) + A_1 f(x_{n+1}, y_{n+1}) + \cdots \\ + A_{n+k-1} f(x_{n+k-1}, y_{n+k-1}) + A_n f(x_{n+k}, y_{n+k}))\,. \end{aligned} \tag{7.34}$$

For notational convenience, we will set $f_n = f(x_n, y_n)$. Deriving these methods is simply a matter of constructing quadrature formulae as we did in the chapter on numerical integration. However care must be taken, because as the example with Simpson's rule shows, not all quadrature schemes lead to stable methods for the numerical solution of IVPs.

An alternative, though equivalent approach to deriving the Adams schemes is based upon expanding the function y in a Taylor series about the initial condition, then using finite differences to approximate the derivatives.

By Taylor's Theorem, we can write

$$y(x+h) = y(x) + y'(x)h + \frac{1}{2}y''(x)h^2 + \frac{1}{6}y'''(x)h^3 + \cdots \tag{7.35}$$

And since $y' = f(x, y(x))$ this implies

$$y_{i+1} = y_i + hf_i + \frac{h^2}{2}f_i^{'} + \frac{h^3}{3!}f_i^{''} + \frac{h^4}{4!}f_i^{'''} + \cdots$$

where the dashes denote differentiation. If we use the approximation $f_i' = \frac{f_i - f_{i-1}}{h} + \frac{h}{2} f_i'' + O(h^2)$ and take the terms up to order h^3 we get

$$y_{i+1} = y_i + h f_i + \frac{h^2}{2} \left[\frac{f_i - f_{i-1}}{h} + \frac{h}{2} f_i'' + O(h^2) \right] + \frac{h^3}{6} f_i'' + O(h^4)$$

So neglecting the terms of order h^2, we get the scheme

$$y_{i+1} = y_i + h \left(\frac{3}{2} f_i - \frac{1}{2} f_{i-1} \right). \tag{7.36}$$

This is the second Adams-Bashforth method. Using higher order Taylor approximations for f', f'' etc, will give higher order explicit Adams-Bashforth schemes. These are also known as Adams-Bashforth open schemes. Let us now list the first five such schemes. Here we have shifted the subscripts in an obvious way for notational convenience.

$$y_{n+1} = y_n + h f_n \tag{7.37}$$

$$y_{n+2} = y_{n+1} + \frac{h}{2} [3 f_{n+1} - f_n] \tag{7.38}$$

$$y_{n+3} = y_{n+2} + \frac{h}{12} [23 f_{n+2} - 16 f_{n+1} + 5 f_n] \tag{7.39}$$

$$y_{n+4} = y_{n+3} + \frac{h}{24} [55 f_{n+3} - 59 f_{n+2} + 37 f_{n+1} - 9 f_n] \tag{7.40}$$

$$y_{n+5} = y_{n+4} + \frac{h}{720} [1901 f_{n+4} - 2774 f_{n+3} + 2616 f_{n+2} - 1274 f_{n+1} + 251 f_n] \tag{7.41}$$

Notice that the coefficients become large very quickly. Also notice that the first Adams-Bashforth method is simply Euler's method.

The obvious question to ask is how do we implement such a scheme? In contrast to one step methods, in order to use a k step method, we require k initial values. If we are to implement say the scheme $y_{n+3} = y_{n+2} + \frac{h}{12}[23 f_{n+2} - 16 f_{n+1} + 5 f_n]$, then we need initial values for y_0, y_1 and y_2. The simplest way to do this is to use a one step method to obtain the necessary starting values, then once the desired starting values are obtained, use the k step method.

Example 7.4. Use the $n = 3$ Adams-Bashforth scheme to solve the ODE $y'(x) = x(1 + y^2),\ y(0) = 0$ on the interval $[0, 1]$. Take $h = 0.1$.

We use the following procedure to find the starting values. From the list of Adams-Bashforth schemes we have

$$\begin{aligned} y_{n+1} &= y_n + h f_n \\ y_{n+2} &= y_{n+1} + \frac{h}{2} [3 f_{n+1} - f_n] \\ y_{n+3} &= y_{n+2} + \frac{h}{12} [23 f_{n+2} - 16 f_{n+1} + 5 f_n] \end{aligned}$$

We use the first scheme to obtain y_1. Then we use the second scheme to obtain y_2 using the Euler approximation for y_1. Thus $y_0 = 0$. From the Euler scheme we obtain

$$y_1 = y_0 + hf(0, y_0) = 0 + 0.1(0(1 + 0^2)) = 0. \tag{7.42}$$

Then

$$y_2 = y_1 + \frac{h}{2}[3f_1 - f_0] = 0 + 0.05(3(0.1(1 + 0^2)) - 0) = 0.015 \tag{7.43}$$

We can now employ the three step Adams-Bashforth scheme to generate the rest of our iterates. For example,

$$\begin{aligned} y_3 &= y_2 + \frac{h}{12}(23f_2 - 16f_1 + 5f_0) \\ &= 0.015 + \frac{0.1}{12}(23(0.2(1 + (0.15)^2)) - 16 \times 0.1 + 5 \times 0) \\ &= 0.0400086 \end{aligned}$$

Continuing in this manner produces the following results.

x_i	Adam-Bashforth y_i	Exact $y(x_i)$	Absolute error
0	0	0	0
0.1	0	0.00500004	0.00500004
0.2	0.015	0.0200027	0.00500267
0.3	0.0400086	0.0450304	0.00502177
0.4	0.0750947	0.0801711	0.00507644
0.5	0.120465	0.125655	0.00519029
0.6	0.176575	0.18197	0.00539471
0.7	0.244287	0.250023	0.00573577
0.8	0.325101	0.331389	0.00628793
0.9	0.421517	0.4287	0.0071824
1	0.537633	0.546302	0.00866929

We see that we have good accuracy, comparable to a second order Taylor scheme. We will get better accuracy if we can find better estimates for y_1 and y_2. For example if we were to take $y_1 = 0.005$ and $y_2 = 0.02$ then we would obtain $y_{10} = 0.544067$ and error of around 0.002.

This naturally brings us to a consideration of the implicit Adams schemes, the so called Adams-Moulton schemes. Their derivation is similar to the derivation of the Adams-Bashforth explicit methods. We now list the first five Adams-Moulton schemes.

$$y_{n+1} = y_n + hf_{n+1} \tag{7.44}$$

$$y_{n+2} = y_{n+1} + \frac{h}{2}[f_{n+2} + f_{n+1}] \tag{7.45}$$

$$y_{n+3} = y_{n+2} + \frac{h}{12}[5f_{n+3} + 8f_{n+2} - f_{n+1}] \tag{7.46}$$

$$y_{n+4} = y_{n+3} + \frac{h}{24}[9f_{n+4} + 19f_{n+3} - 5f_{n+2} + f_{n+1}] \tag{7.47}$$

$$y_{n+5} = y_{n+4} + \frac{h}{720}]251f_{n+5} + 646f_{n+4} - 264f_{n+3} + 106f_{n+2} - 19f_{n+1}] \tag{7.48}$$

Notice that the first equation is similar to Euler's method. It is known as a backward Euler's method. Also notice that in each scheme, the next term y_{n+k} occurs on both sides of the equation. So we do not obtain it directly. Rather we have to solve for it.

To illustrate. In our previous example, we had $y' = x(1 + y^2)$. If we take the second Adams-Moulton scheme, then given y_{n+1} we obtain y_{n+2} by solving the equation

$$y_{n+2} = y_{n+1} + \frac{h}{2}[x_{n+2}(1 + y_{n+2}^2) + x_{n+1}(1 + y_{n+1}^2)] \tag{7.49}$$

This is a quadratic, so we could employ the quadratic formula to obtain y_{n+2}. However for a general IVP, there will be no easy way of solving for y_{n+2}. If we are to use an implicit scheme by itself, we must employ some kind of numerical method for solving the resulting equation. Newton's method is a good choice.

However, Adams-Moulton methods and Adams-Bashforth methods are rarely used by themselves. Instead, they are normally used in pairs, with an explicit scheme combined with an implicit scheme. We illustrate the procedure by an example.

Consider the third order Adams schemes, and assume we have obtained starting values y_0, y_1 and y_2. We use the explicit scheme to produce an estimate for y_{n+3} which we call $\tilde{y}_{n+3}$. The estimate is given by

$$\tilde{y}_{n+3} = y_{n+2} + \frac{h}{12}\left[23f_{n+2} - 16f_{n+1} + 5f_n\right] \tag{7.50}$$

Now we take the corresponding implicit scheme. Instead of solving this for y_{n+3} we use our estimate $\tilde{y}_{n+3}$ from the explicit scheme to produce a new estimate for y_{n+3}. That is, in order to produce y_{n+3} we calculate

$$y_{n+3} = y_{n+2} + \frac{h}{12}[5f(x_{n+3}, \tilde{y}_{n+3}) + 8f_{n+2} - f_{n+1}] \tag{7.51}$$

Thus calculating y_{n+3} is done in two steps. Equation (7.50) is known as the *predictor* and (7.51) is known as the *corrector*.

One common practice which can be used to refine this method even further, is to add an iterative procedure involving the corrector. What this means is that, having obtained an estimate y^1_{n+3} by applying the corrector to $\tilde{y}_{n+3}$, we then obtain a new estimate y^2_{n+3} by repeating the process.

More precisely, we take $\tilde{y}_{n+3}$ from the predictor. We then produce

$$y^1_{n+3} = y_{n+2} + \frac{h}{12}[5f(x_{n+3}, \tilde{y}_{n+3}) + 8f_{n+2} - f_{n+1}].$$

Then y^2_{n+3} is given by

$$y^2_{n+3} = y_{n+2} + \frac{h}{12}[5f(x_{n+3}, y^1_{n+3}) + 8f_{n+2} - f_{n+1}].$$

In general we form the sequence

$$y^{k+1}_{n+3} = y_{n+2} + \frac{h}{12}[5f(x_{n+3}, y^k_{n+3}) + 8f_{n+2} - f_{n+1}].$$

We keep iterating until we have convergence. This limit is taken to be the estimate for y_{n+3}. Then we move back to the predictor and produce and estimate $\tilde{y}_{n+4}$, for the next term y_{n+4} and repeat.

Although this can be computationally intensive, it is also extremely effective. Predictor-Corrector methods are among the most successful schemes that we have for solving IVPs. The degree of computational effort we use depends as always on the accuracy which we desire. For example, the iterative procedure described above, where we produce a sequence y^k_{n+3} by iterating the corrector, is often omitted. Usually the estimate for y_{n+3} produced by (7.51) is good enough for most purposes.

Example 7.5. Use the $n = 3$ Adams-Bashforth, Adams-Moulton predictor-corrector scheme to solve the ODE $y'(x) = x(1+y^2)$, $y(0) = 0$ on the interval $[0, 1]$. Take $h = 0.1$.

We take the same starting values as before. We obtained in the previous example an estimate for y_3. This is our value $\tilde{y}_3$. That is

$$\begin{aligned}
\tilde{y}_3 &= y_2 + \frac{h}{12}(23f_2 - 16f_1 + 5f_0) \\
&= 0.015 + \frac{0.1}{12}(23(0.2(1 + (0.15)^2)) - 16 \times 0.1 + 5 \times 0) \\
&= 0.0400086
\end{aligned}$$

Whence

$$y_3 = y_2 + \frac{h}{12}(5f(x_3, \tilde{y}_3) + 8f(x_2, y_2) - f(x_1, y_1))$$
$$= 0.015 + \frac{0.1}{12}(5 \times 0.3(1 + (0.0400086)^2 + 8 \times 0.2(1 + 0.15^2)$$
$$- 0.1(1 + 0^2)) = 0.040023.$$

Continuing produces the following results.

x_i	Predictor-Corrector y_i	Exact $y(x_i)$	Absolute error
0	0	0	0
0.1	0	0.00500004	0.00500004
0.2	0.015	0.0200027	0.00500267
0.3	0.040023	0.0450304	0.00500739
0.4	0.0751487	0.0801711	0.00502241
0.5	0.120598	0.125655	0.00505725
0.6	0.176845	0.18197	0.00512501
0.7	0.244779	0.250023	0.00524326
0.8	0.325953	0.331389	0.00543598
0.9	0.422962	0.4287	0.00573728
1	0.540103	0.546302	0.00619905

It is apparent that this is somewhat more accurate than just the $n = 3$ Adams-Bashforth scheme on its own, though the improvement is not great. We could again achieve considerably greater accuracy by taking more care to calculate the starting values. For example, starting with $y_1 = 0.005, y_2 = 0.02$ gives an estimate of $y_{10} = 0.54658$ An error of approximately 0.0003. Iterative refinement will also improve the accuracy of the solution.

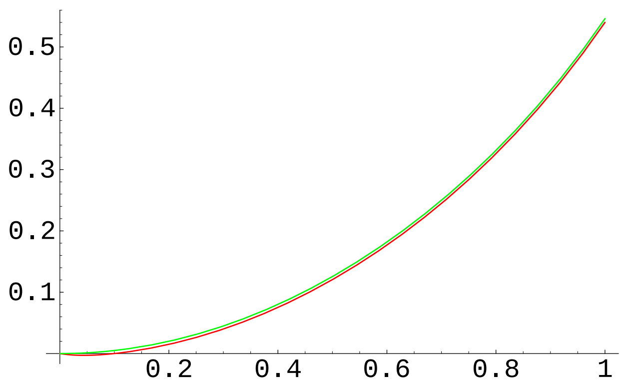

Figure 16. Predictor-Corrector versus true solution.

The graph shows the predictor-corrector solution plotted against the true solution. Notice that the two solutions are extremely close.

7.6. **Finite Difference Methods.** We come now to a particularly important class of methods for the numerical solution of differential equations. The so called finite difference methods. These methods are particularly useful for the numerical solution of *boundary value problems* for second and higher order equations. They also provide one of the most widely used techniques for the numerical solution of partial differential equations.

In this section we will concentrate on the use of finite difference methods for boundary value problems. To begin, we will define what these problems are.

Definition 7.1. Let

$$f(x, y', ..., y^n) = 0 \tag{7.52}$$

be an nth order, ordinary differential equation defined on the interval $I = [a, b]$. A *boundary value problem* (BVP) for (7.52) is defined to be the problem of finding a solution of (7.52) which satisfies the *boundary conditions* $y(a) = \alpha$ anf $y(b) = \beta$.

This formulation of a BVP is actually a simplified version of the general problem. In many problems, the boundary conditions are given in more complicated form. For example, we might have the conditions given as $k_1y(a)+k_2y(b) = \alpha$ and $k_3y(a)+k_4y(b) = \beta$. Or the boundary condition might be given in terms of the derivatives as well. For example $k_1y'(a) + k_2y(b) = \alpha$ and $k_3y(a) + k_4y'(b) = \beta$. Many formulations are possible. We will concentrate on those problems encompassed by Definition 1.1.

Under certain circumstances, it is possible to prove the existence and uniqueness of a solution to a BVP. One such theorem is given next.

Theorem 7.2. *Consider the boundary value problem*

$$-y''(x) + r(x)y(x) = \varphi(x), \qquad a \le x \le b, \tag{7.53}$$

$$y(a) = y(b) = 0 \tag{7.54}$$

where $r, \varphi : [a, b] \to \mathbb{R}$ *are continuous functions. The BVP (7.53) has a unique solution, y which is at least twice differentiable on* $[a, b]$*, if* $r(x) \ge 0$*, for* $x \in [a, b]$.

Many BVPs can be put into this form by a change of variable, making the theorem more general than it appears. Boundary value problems of more general nature can also be treated.

Let us now solve a simple BVP.

Example 7.6. Solve the BVP

$$y'' + y = 0,\ y(0) = 0,\ y(\frac{\pi}{2}) = 1.$$

We first obtain the solution of the equation $y'' + y = 0$. The characteristic equation is $\lambda^2 + 1 = 0$ which has roots $\pm i$. Hence the general solution is

$$y = A\cos x + B\sin x$$

for constants A and B.

To solve the BVP, we have to fit the boundary conditions. The solution must satisfy $y(0) = 0$ and $y(\frac{\pi}{2}) = 1$. Hence

$$A\cos 0 + B\sin 0 = A = 0.$$

Next,

$$y(\frac{\pi}{2}) = B\sin\frac{\pi}{2} = B = 1.$$

So the solution of the boundary value problem is $y = \sin x$.

Notice something interesting about this problem. If instead of choosing the interval $[0, \frac{\pi}{2}]$, we had chosen the interval $[0, \pi]$ then the resulting BVP, with y having to satisfy $y(\pi) = 1$, would have no solution. Why? Because the final condition we have to fit would become

$$y(\pi) = B\sin\pi = 0 = 1.$$

So no solution exists. It is also possible to make a choice of boundary values such that the problem has infinitely many solutions. For infinitely many solutions we would set $y(\pi) = 0$.

Given a BVP, which has a unique solution, we now ask how we can solve it numerically? It is clear that a different approach is needed than for solving IVPs. With an IVP, we can start with a value, and simply allow the step the solution y forward in time using the information we have about the derivatives of y. However, this will not work (at least not without major modification) for a BVP. Because we need the solution to not only take a given value at $x = a$, we also require the solution to take a given value at $x = b$. It is not obvious that the methods we have already developed will guarantee this. For example, say we want to solve $y' = g(x, y(x))$ subject to $y(a) = y_0$ and $y(b) = y_1$. We might try Euler's method. We choose an h and produce iterates by calculating $y^{i+1} = y^i + hg(x_i, y^i)$ starting with $y^0 = y(a)$. However doing this is unlikely to work. We would have to be very lucky for the approximate value of y at $x = b$ given by Euler's method, to be anything like the true value.

So we require another approach. One very powerful method of solving such problems is the method of finite differences. The idea is essentially this. We replace the differential equation with a difference equation. There are many ways in which this can be done.

We make the following observation. If $y(x)$ is twice differentiable and h is small, then we can approximate the first derivative of y by a finite difference:

$$y'(x_i) \approx \frac{y(x_i + h) - y(x_i)}{h}. \tag{7.55}$$

This however, is not the only possible approximation. We could also use the *central difference* approximation

$$y'(x_i) \approx \frac{y(x_i+h)-y(x_i-h)}{2h}. \tag{7.56}$$

This central difference approximation to the derivative is extremely useful. When solving a BVP with a first derivative term, the central difference approximation usually gives better results than using (7.55).

Moreover, we can approximate the second derivative by a finite difference too. We have

$$y''(x_i) \approx \frac{y(x_i+h)-2y(x_i)+y(x_i-h)}{h^2}. \tag{7.57}$$

Higher order derivatives can be approximated in the same manner.

Now let us introduce some helpful notation. We are interested in solving a BVP on the interval $[a,b]$. So we will let $x_0 = a$ We divide $[a,b]$ up into n equal subintervals of length h. So that

$$x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, ..., x_n = x_0 + nh = b.$$

The idea is to attempt to use finite difference methods to approximate the value of the BVP at the discrete points $x_0, ..., x_n$. We let the value of the approximations at x_i be denoted y_i.

Now suppose that we wish to solve the following second order BVP.

$$y'' = f(x,y,y'),\ y(a) = \alpha,\ y(b) = \beta. \tag{7.58}$$

The finite difference approximation to this BVP is

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2} = f\left(x_i, y_i, \frac{y_{i+1}-y_{i-1}}{2h}\right) \tag{7.59}$$

$$i = 1, 2, ..., n-1,\ y_0 = \alpha,\ y_n = \beta.$$

This gives us a system of equations which we have to solve for the approximations y_i. Notice that if the function f is nonlinear in y and/or y' then the equations we must solve for the y_i are nonlinear.

We now consider an example.

Example 7.7. Solve the problem in example 10.1 by using the finite difference approximation (7.59).

The equation we wish to solve is $y''+y = 0$. Thus the finite difference approximation is

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2} + y_i = 0 \tag{7.60}$$

with $y_0 = 0, y_n = 1$. Multiplying through by h^2, we get the system of linear equations

$$y_{i+1} - 2y_i + y_{i-1} + h^2 y_i = y_{i+1} + (h^2-2)y_i + y_{i-1} = 0, i = 1, ..., n-1$$

We have the further conditions that $y_0 = 0,\ y_n = 1$. We have to incorporate this into the equations before we can solve them.

Starting with $i = 1$ the system can be written

$$\begin{aligned} y_2 + (h^2 - 2)y_1 + y_0 &= 0 \\ y_3 + (h^2 - 2)y_2 + y_1 &= 0 \\ &\vdots \\ y_n + (h^2 - 2)y_{n-1} + y_{n-2} &= 0 \end{aligned}$$

However, because $y_0 = 0$ and $y_n = 1$, the equations simplify to

$$\begin{aligned} y_2 + (h^2 - 2)y_1 &= 0 \\ y_3 + (h^2 - 2)y_2 + y_1 &= 0 \\ &\vdots \\ (h^2 - 2)y_{n-1} + y_{n-2} &= -1 \end{aligned}$$

We thus have an $(n-1) \times (n-1)$ system of linear equations in $n-1$ unknowns. In matrix form we can write it as

$$\begin{pmatrix} h^2-2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & h^2-2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & h^2-2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & h^2-2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$$

If we let

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}, \ \tilde{\mathbf{y}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$$

then we express the system in more compact form as

$$A\mathbf{y} = \tilde{\mathbf{y}}.$$

The matrix A is tridiagonal. If we choose n we can now solve this system for the values of y_i. Let us take $n = 10$. Then $h = \frac{\pi}{20}$. The system of equations we have to solve is then the 9×9 system

$$\begin{pmatrix} -1.97533 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -1.97533 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -1.97533 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -1.97533 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$$

We now have the problem of how to solve this system. We can employ any of the methods we know from linear algebra. For simplicity, we present here the solution obtained from using the LinearSolve command in Mathematica. Let us compare the numerical solution by finite differences with the true solution found previously to be $y = \sin x$.

n	FD y_i	Exact $y(x_i)$	Absolute error
0	0	0	0
1	0.156595	0.156434	0.0002
2	0.309325	0.309016	0.0003
3	0.454424	0.453991	0.0004
4	0.58831	0.587785	0.0005
5	0.70768	0.707106	0.0005
6	0.809589	0.809017	0.0006
7	0.891522	0.891006	0.0005
8	0.951457	0.951057	0.0004
9	0.987917	0.987688	0.0002
10	1	1	0

Clearly the numerical approximation is good. We have a maximum error of about 0.0005 in absolute terms. To get a better approximation, we would need to take a larger value of n.

7.7. **Inhomogeneous Boundary Value Problems.** It is a simple matter to use the method of finite differences to solve inhomogeneous problems. Again we will illustrate by example.

Example 7.8. Solve the BVP

$$y'' + 3y' + 2y = x^2,\ y(0) = 1,\ y(1) = 2,$$

by finite differences.

First we obtain the exact solution of the problem. First we solve the homogeneous problem

$$y'' + 3y' + 2y = 0.$$

The characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$. This leads to the homogeneous solution $y_h = Ae^{-2x} + Be^{-x}$. To find a particular integral, we can look for a solution of the form $y_p = ax^2 + bx + c$. Substitution of y_p into the equation produces the general solution to the equation

$$y = Ae^{-2x} + Be^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}.$$

We now fit the boundary conditions by solving for A and B.

$$A + B + \frac{7}{4} = 1 \tag{7.61}$$

$$Ae^{-2} + Be^{-1} + \frac{1}{2} - \frac{3}{2} + \frac{7}{4} = 2. \tag{7.62}$$

This gives

$$y = -\frac{e(3+5e)}{4(e-1)}e^{-2x} + \frac{3+5e^2}{4(e-1)}e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}.$$

Now let us solve the BVP numerically using finite difference methods.

As before we use

$$y'(x_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}.$$

$$y''(x_i) \approx \frac{y(x_{i+1} - 2y_i + y_{i-1}}{h^2}.$$

The equation thus becomes

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 3\frac{y_{i+1} - y_{i-1}}{2h} + 2y_i = x_i^2,\ i = 1, ..., n-1.$$

Multiplying through by h^2, gives the system of equations

$$y_{i+1} - 2y_i + y_{i-1} + \frac{3}{2}h(y_{i+1} - y_{i-1}) + 2h^2 y_i = x_i^2 h^2,\ i = 1, ..., n-1.$$

Collecting terms, we have

$$(1 + \frac{3}{2}h)y_{i+1} + 2(h^2 - 1)y_i + (1 - \frac{3}{2}h)y_{i-1} = (x_i h)^2,\ i = 1, ..., n-1. \tag{7.63}$$

Where $y_0 = 1,\ y_n = 2$. From this point the solution proceeds exactly as in the previous example. We write this system out and obtain

$$\begin{aligned}
(1 + \frac{3}{2}h)y_2 + 2(h^2 - 1)y_1 + (1 - \frac{3}{2}h)y_0 &= (x_1 h)^2 \\
(1 + \frac{3}{2}h)y_3 + 2(h^2 - 1)y_2 + (1 - \frac{3}{2}h)y_1 &= (x_2 h)^2 \\
\vdots \qquad\qquad &\qquad \vdots \\
(1 + \frac{3}{2}h)y_n + 2(h^2 - 1)y_{n-1} + (1 - \frac{3}{2}h)y_{n-2} &= (x_{n-1} h)^2
\end{aligned}$$

Since $y_0 = 1$ and $y_n = 2$ this is equivalent to

$$\begin{aligned}
(1 + \frac{3}{2}h)y_2 + 2(h^2 - 1)y_1 &= (x_1 h)^2 - (1 - \frac{3}{2}h) \\
(1 + \frac{3}{2}h)y_3 + 2(h^2 - 1)y_2 + (1 - \frac{3}{2}h)y_1 &= (x_2 h)^2 \\
(1 + \frac{3}{2}h)y_4 + 2(h^2 - 1)y_3 + (1 - \frac{3}{2}h)y_2 &= (x_3 h)^2 \\
\vdots \qquad\qquad &\qquad \vdots \\
(1 + \frac{3}{2}h)y_{n-1} + 2(h^2 - 1)y_{n-2} + (1 - \frac{3}{2}h)y_{n-3} &= (x_{n-2} h)^2 \\
2(h^2 - 1)y_{n-1} + (1 - \frac{3}{2}h)y_{n-2} &= (x_{n-1} h)^2 - 2(1 + \frac{3}{2}h)
\end{aligned}$$

In matrix form, this is $A\mathbf{y} = \tilde{\mathbf{y}}$ where

$$A = \begin{pmatrix} 2(h^2-1) & 1+\frac{3}{2}h & 0 & \cdots & \cdots & 0 \\ 1-\frac{3}{2}h & 2(h^2-1) & 1+\frac{3}{2}h & 0 & \cdots & 0 \\ 0 & 1-\frac{3}{2}h & 2(h^2-1) & 1+\frac{3}{2}h & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 1-\frac{3}{2}h & 2(h^2-1) \end{pmatrix}$$

and

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}, \ \tilde{\mathbf{y}} = \begin{pmatrix} (x_1h)^2 - (1-\frac{3}{2}h) \\ (x_2h)^2 \\ (x_3h)^2 \\ \vdots \\ (x_{n-1}h)^2 - 2(1+\frac{3}{2}h) \end{pmatrix}.$$

Again the matrix is tridiagonal. So to solve the BVP, we must choose a value of h. The smaller h is the more accurate the solution should be. However, since $h = (b-a)/n$ making h smaller increases the size of the linear system which we must solve. We will take $n = 10$, which gives $h = 0.1$. For this choice of h, A is the 9×9 matrix

$$A = \begin{pmatrix} -1.98 & 1.15 & 0 & \cdots & \cdots & 0 \\ 0.85 & -1.98 & 1.15 & 0 & \cdots & 0 \\ 0 & 0.85 & -1.98 & 1.15 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0.85 & -1.98 & 1.15 \\ \\ 0 & 0 & \cdots & 0 & 0.85 & -1.98 \end{pmatrix},$$

and

$$\tilde{\mathbf{y}} = \begin{pmatrix} -0.8499 \\ 0.0004 \\ 0.0009 \\ 0.0016 \\ 0.0025 \\ 0.0036 \\ 0.0049 \\ 0.0064 \\ -2.2919 \end{pmatrix}.$$

Again we use the LinearSolve command in Mathematica to invert the system. The results are as follows.

n	FD y_i	Exact $y(x_i)$	Absolute error
0	1	1	0
1	1.4959	1.49139	0.005
2	1.83651	1.82979	0.006
3	2.05668	2.0493	0.007

n	FD y_i	Exact $y(x_i)$	Absolute error
4	2.18442	2.17736	0.007
5	2.24224	2.23608	0.006
6	2.24816	2.24321	0.005
7	2.21656	2.21294	0.003
8	2.15892	2.15661	0.002
9	2.08433	2.08324	0.001
10	2	2	0

Again the results are good. The error is greatest in the middle of the interval and decreases towards the end. We expect this because fitting the boundary conditions forces the error to be zero at the endpoints.

The theory of finite difference methods is very well developed and new methods are being produced all the time. Clearly in an introductory treatment we can do no more than present the basics. One of the major questions is that of obtaining error estimates for our numerical solutions. A good deal of work has been done on this problem. We shall content ourselves with stating an important theorem.

Theorem 7.3. *Let the boundary value problem (7.53) have a unique solution y. Assume further that y is four times differentiable on $[a, b]$ with fourth derivative $y^{(4)}$. Let y_k be the finite difference approximation to y at $x_k \in [a, b]$, obtained by taking $h = (b-a)/n$. Then the following error estimate holds*

$$\max_{0 \le k \le n} |y_k - y(x_k)| \le Mh^2, \tag{7.64}$$

where $M = \frac{(b-a)^2}{96} \|y^{(4)}\|_\infty$.

By making further assumptions on the smoothness of the solutions of the BVP, better estimates can be established. However, this result is quite a strong one. It tells us that the error in the finite difference estimate for y at x_k is proportional to h^2. This means that if we let $h \to 0$ then the error in the finite difference estimate converges to zero. Thus we can obtain as precise a numerical estimate as we desire, by making h sufficiently small. The drawback of course, is that the smaller h is, the larger n is, and consequently, the larger the linear system which we have to solve.

Nevertheless, finite difference methods have been extremely effective in the solution of differential equations and they are essential tools in the analysis of a vast range of problems. It is possible to extend these methods to the solution of partial differential equations, but that is beyond the scope of the course.

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