All Working Must Be Shown For Every Question.

Question 1 (2 + 5 + 3= 10 marks).

(1) The ordinary differential equation

$$x^2y'' + 3xy' - 3y = 0,$$

has a solution $y_1(x) = x$. Use this to find a second linearly independent solution y_2.

(2) Use variation of parameters to find the general solution of the ODE

$$x^2y'' + 3xy' - 3y = x^2e^x.$$

(3) Obtain the solution of the ordinary differential equation

$$x^2y'' + 3xy' + (1 + 4x^4) = 0,$$

in terms of Bessel functions.

OVER PAGE

Question 2 (10 marks).

(a) Obtain the general solution of the ODE
$$y'' + xy' + 2y = 0,$$
by letting $y = \displaystyle\sum_{n=0}^{\infty} a_n x^n$.

Question 3 (10 marks).

(1) Find a solution of the equation

$$2xy'' + (1+x)y' + y = 0,$$

of the form $y = \sum_{n=0}^{\infty} a_n x^{n+s}$.

Show that the exponents are $s_1 = 0, s_2 = 1/2$ and that a_n satisfies

$$a_{n+1} = \frac{-a_n}{(2n+2s+1)}.$$

Find the solution corresponding to both exponents.

Question 4 (7+3=10 marks)

(i) Use the Laplace transform to solve the initial value problem

$$y'' + 4y = e^{-x}, \tag{0.1}$$

$y(0) = 0, y'(0) = 0$. Note that we can write

$$\frac{1}{(s+a)(s^2+b^2)} = \frac{A}{s+a} + \frac{Bs+C}{s^2+b^2},$$

for certain constants A, B, C.

(ii) Obtain the inverse Laplace transform of

$$F(s) = \frac{1}{s^2}\frac{1}{\sqrt{s^2+1}},$$

as a convolution. The transforms you need are in the table at the back of the exam.

OVER PAGE

Question 5 (3+7=10 marks).

(i) Calculate the Fourier sine series for the function

$$f(x) = x(a - x), \ 0 \leq x < a.$$

(ii) Consider the following boundary value problem for the equation.

$$u_{xx} + 4u_{yy} = 0, \ 0 \leq x \leq 1, 0 \leq y \leq 1,$$
$$u(x, 0) = f(x), \ u(x, 1) = 0, \ u(0, y) = 0, u(1, y) = 0.$$

By looking for solutions of the form $u(x, y) = X(x)Y(y)$, show that

$$X''(x) = \lambda X(x), \quad X(0) = X(1) = 0,$$

for some constant λ and determine the value of λ. Thus obtain a formula for X. Then show that

$$Y(y) = C \cosh\left(\frac{n\pi y}{2}\right) + D \sinh\left(\frac{n\pi y}{2}\right).$$

Use the fact that $Y(1) = 0$ and the identity

$$\sinh(a - b) = \sinh a \cosh b - \sinh b \cosh a$$

to simplify the expression for Y.
Hence write down the solution of the problem in part (ii) as an infinite series.

Table of integrals

$$\int u^n\,du = \frac{u^{n+1}}{n+1}$$

$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u$$

$$\int \frac{du}{u} = \log|u|$$

$$\int \cos u\,du = \sin u$$

$$\int \frac{du}{1-u^2} = \tanh^{-1} u$$

$$\int \sin u\,du = -\cos u$$

$$\int x^2\sin(ax)dx = \frac{(2-a^2x^2)\cos(ax)+2ax\sin(ax)}{a^3}$$

$$\int \sec^2 u\,du = \tan u$$

$$\int \cosh u\,du = \sinh u$$

$$\int \csc^2 u\,du = -\cot u$$

$$\int \sinh u\,du = \cosh u$$

$$\int \sec u\tan u\,du = \sec u$$

$$\int \tanh u\,du = \log\cosh u$$

$$\int \csc u\cot u\,du = -\csc u$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}$$

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx.$$

Table of Laplace transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$
$$\mathcal{L}(e^{-at}) = \frac{1}{s+a}$$
$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$$
$$\mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2}$$
$$\mathcal{L}(J_0(t)) = \frac{1}{\sqrt{1+s^2}}.$$

Variation of parameters

Given that y_1 and y_2 are solutions of the ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0,$$

we seek a particular solution of the ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x),$$

by looking for solutions of the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$. The functions u and v must satisfy

$$u'y_1 + v'y_2 = 0,$$
$$u'y_1' + v'y_2' = f(x).$$

Bessel Functions

Bessel's differential equation $t^2u'' + tu' + (t^2 - \alpha^2)u = 0$ may be transformed into the equation

$$x^2y'' + (1-2s)xy' + ((s^2 - r^2\alpha^2) + a^2r^2x^{2r})y = 0$$

under the change of variables $t = ax^r$ and $y(x) = x^s u(t)$.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$