Example Solve $y'' + a^2 y = f(t)$, $y(0) = 0$, $y'(0) = 1$. Take L.T. of both sides.

$Y = \mathcal{L}y$, $F = \mathcal{L}f$

$$\mathcal{L}(y'') + a^2 \mathcal{L}(y) = \mathcal{L}f$$

Hence $s^2 Y(s) - s y(0) - y'(0) + a^2 Y = F$

$$\therefore (s^2 + a^2) Y(s) - 1 = F(s)$$

$$Y(s) = \frac{1}{s^2+a^2} + \frac{F(s)}{s^2+a^2}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] + \mathcal{L}^{-1}\left[\frac{F(s)}{s^2+a^2}\right]$$

$$= \frac{1}{a}\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] + \frac{1}{a}\mathcal{L}^{-1}\left[\frac{a \cdot F(s)}{s^2+a^2}\right]$$

$$= \frac{1}{a}\sin(at) + \frac{1}{a}\int_0^t f(u)\sin(a(t-u))\,du$$

Example Solve $ty'' + y = 0$, $y(0) = 0$.

Let $Y(s) = \mathcal{L}y$. Recall $\mathcal{L}(t f(t)) = -\frac{d}{ds}F(s)$

$F = \mathcal{L}f$

So $\mathcal{L}(t y'') = -\frac{d}{ds}\mathcal{L}(y'')$

$$= -\frac{d}{ds}\left(s^2 Y(s) - s y(0) - y'(0)\right)$$

$$= -s^2 \frac{dY}{ds} - 2sY(s) + y(0)$$

$$-s^2\frac{dY}{ds} - 2sY(s) + Y(s) = 0$$

$$\Longrightarrow s^2 \frac{dY}{ds} + (2s-1)Y = 0$$

$\therefore\ s^2\frac{dY}{ds} = -(2s-1)Y(s)$. First order separable

$$\Rightarrow \frac{dY}{Y} = -\left(\frac{2s-1}{s^2}\right)ds \quad . \text{ So we have}$$

$$\int \frac{dY}{Y} = -\int\left(\frac{2}{s} - \frac{1}{s^2}\right)ds$$

Thus $\ln Y = -2\ln s - \frac{1}{s} + C$ Exponentiating gives

$$Y(s) = \frac{A}{s^2}e^{-1/s}, \qquad A = e^{c},$$

$$= A\left(\frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{2!\,s^4} - \frac{1}{3!\,s^5} + \cdots\right)$$

$$y(t) = A\mathcal{L}^{-1}\left(\frac{1}{s^2}e^{-1/s}\right) = A\left(\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) + \frac{1}{2!}\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) - \frac{1}{3!}\mathcal{L}^{-1}\left(\frac{1}{s^5}\right) + \cdots\right)$$

$$= A\left(t - \frac{1}{2!}t^2 + \frac{1}{2!\,3!}t^3 - \frac{1}{3!\,4!}t^4 + \cdots\right) \quad \left[\mathcal{L}t^n = \frac{n!}{s^{n+1}}\right]$$

$$= A\sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^n}{(n-1)!\,n!} = At\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!\,(n+1)!}$$

Now

$$J_a(t) = \left(\frac{t}{2}\right)^a\sum_{n=0}^{\infty}(-1)^n\frac{(t/2)^{2n}}{n!\,\Gamma(n+a+1)}$$

Take $a = 1$

$$J_1(t) = \frac{t}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(t/2)^{2n}}{n!\,(n+1)!}$$

Now we have

$$J_1(2\sqrt{t}) = \sqrt{t}\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!\,(n+1)!}.$$

Comparing we see that

$$y(t) = At\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!\,(n+1)!}$$

$$= A\sqrt{t}\,J_1(2\sqrt{t}).$$

Notice that the gen solution of

$$ty'' + y = 0$$

$$y = c_1 t^{1/2} J_1(2\sqrt{t}) + c_2 t^{1/2} Y_1(2\sqrt{t})$$

using our previous methods.

In fact $\mathcal{L}^{-1}\left(\frac{1}{s^n} e^{-\frac{a}{s}}\right) = \left(\frac{a}{t}\right)^{\frac{1-n}{2}} J_{n-1}(2\sqrt{at})$;

$a > 0$

Example $\quad xy'' + (b-x)y' - ay = 0$

We let

$$y(x) = \int_0^\infty h(t) e^{-xt}\, dt. \text{ Then}$$

$$y' = \int_0^\infty -t h(t) e^{-xt}\, dt$$

$$y'' = \int_0^\infty t^2 h(t) e^{-xt}\, dt \quad \text{and giving}$$

$$xy'' + (b-x)y' - ay = \int_0^\infty x t^2 h(t) e^{-xt}\, dt - (b-x)\int_0^\infty t h(t) e^{-xt}\, dt - \int_0^\infty a h(t) e^{-xt}\, dt$$

$$= -\int_0^\infty t^2 h(t) \frac{d}{dt}\left(e^{-xt}\right) dt - \int_0^\infty b t h(t) e^{-xt}\, dt - \int_0^\infty t h(t) \frac{d}{dt}\left(e^{-xt}\right) - \int_0^\infty a h(t) e^{-xt}\, dt$$

$$= \cancel{-t^2 h(t) e^{-xt}\Big|_0^\infty}^{\,0} + \int_0^\infty \frac{d}{dt}\left(t^2 h(t)\right) e^{-xt}\, dt - \int_0^\infty b t h(t) e^{-xt}\, dt - \cancel{t h(t) e^{-xt}\Big|_0^\infty}^{\,0} + \int_0^\infty \frac{d}{dt}\left(t h(t)\right) e^{-xt}\, dt - \int_0^\infty a h(t) e^{-xt}\, dt$$

$$= \int_0^\infty \left[t(1+t)h'(t) + (1 - a + (2-b)t)h(t)\right] e^{-xt}\, dt = 0$$

So $t(1+t)h'(t) + (1-a+(2-b)t)h(t) = 0$

This gives

$$h(t) = A t^{a-1}(1+t)^{b-a-1}$$

$$\therefore \quad y(x) = A\int_0^\infty t^{a-1}(1+t)^{b-a-1} e^{-xt}\,dt$$

is a solution

Typically $A = \dfrac{1}{\Gamma(a)}$.

$$U(a,b,x) = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{b-a-1} e^{-xt}\,dt$$

This is

Tricomi's confluent hypergeometric function. We know a second solution is ${}_1F_1(a,b,x) = \sum_{n=0}^\infty \frac{(a)_n}{n!\,(b)_n} x^n$. Kummer's con. hyp. fun.

We can show

$${}_1F_1(a,b,x) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 e^{xt} t^{a-1}(1-t)^{b-a-1}\,dt$$

Many special cases:

$${}_1F_1(a,a,x) = \sum_{n=0}^\infty \frac{1}{n!}\frac{(a)_n}{(a)_n}x^n = e^x.$$

$${}_1F_1(\alpha+\tfrac{1}{2}, 2\alpha+1, 2ix) = \Gamma(\alpha+1)e^{ix}\left(\frac{x}{2}\right)^{-\alpha} J_\alpha(x) \text{ etc.}$$

If $p \le q$

$${}_pF_q(a_1,\dots,a_p, b_1,\dots,b_q, x) = \sum_{n=0}^\infty \frac{1}{n!}\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} x^n$$

— The general hypergeometric function

Partial Differential Equations (PDEs) have more than one independent variable

Example (1) $\dfrac{1}{k}\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, k a constant, $t>0$, $x \in \mathbb{R}$

The heat equation.

(2) $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ $x \in \mathbb{R}$, $t > 0$
c is a constant
The wave equation.

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Laplace's equation.

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}$$

In n dimensions, $\frac{1}{k}\frac{\partial u}{\partial t} = \Delta u$, $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \Delta u$

$\Delta u = 0$

We will solve some PDEs by L.T.

Example $x\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x$, $x, t > 0$

$$u(x,0) = 0, \quad u(0,t) = 0$$

Let $U(x,s) = \int_0^\infty u(x,t)e^{-st}\,dt$

$$\mathcal{L}\left(\frac{\partial u}{\partial t}\right) = \int_0^\infty \frac{\partial u}{\partial t} e^{-st}\,dt = \overset{0}{\left. u(x,t)e^{-st}\right]_0^\infty} + s\int_0^\infty u(x,t)e^{-st}\,dt$$

$$= sU(x,s)$$

$$\mathcal{L}\left(\frac{\partial u}{\partial x}\right) = \int_0^\infty \frac{\partial u}{\partial x} e^{-st}\,dt = \frac{\partial}{\partial x}\int_0^\infty u e^{-st}\,dt$$

$$= \frac{\partial U}{\partial x}$$

$\mathcal{L}(x) = \frac{x}{s}$. $\mathcal{L}\left(x\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right) = \mathcal{L}(x)$

So $xsU + \frac{d}{dx}U = \frac{x}{s}$

Integrating factor is $e^{\int xs\,dx} = e^{s\frac{x^2}{2}}$

$$e^{sx^2/2}xsU + e^{sx^2/2}\frac{dU}{dx} = \frac{x}{s}e^{\frac{sx^2}{2}}.$$

$$\frac{d}{dx}\left(e^{s\frac{x^2}{2}}U\right) = \frac{x}{s}e^{s\frac{x^2}{2}}$$

$$e^{s\frac{x^2}{2}}U = \frac{1}{s}\int x e^{s\frac{x^2}{2}}\,dx \qquad \frac{x^2}{2} = y$$

$$dy = x\,dx$$

$$= \frac{1}{s}\int e^{sy}\,dy$$

$$= \frac{1}{s^2}e^{s\frac{x^2}{2}} + C$$

$$U = \frac{1}{s^2} + Ce^{-sx^2/2}$$

$$U(0,\ ) = \int_0^\infty u(0,t)e^{-st}\,dt = 0$$

Thus $U(0,s) = \frac{1}{s^2} + C = 0$. $\therefore C = -\frac{1}{s^2}$

$$U(x,s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-sx^2/2}.$$

$$u(x,t) = \mathcal{L}^{-1}\left(\frac{1}{s^2} - \frac{1}{s^2}e^{-s\frac{x^2}{2}}\right)$$

$$= t - \mathcal{L}^{-1}\left(\frac{1}{s^2}e^{-sx^2/2}\right)$$

Now

$$\mathcal{L}^{-1}\left(F(s)e^{-sa}\right) = f(t-a)H(t-a)$$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2}e^{-sx^2/2}\right) = \left(t-\frac{x^2}{2}\right)H\left(t-\frac{x^2}{2}\right)$$

$$\therefore u(x,t) = t - \left(t-\frac{x^2}{2}\right)H\left(t-\frac{x^2}{2}\right)$$

$$= \begin{cases} \frac{x^2}{2} & t > \frac{x^2}{2} \\ t & t \le x^2/2 \end{cases}$$