Solutions Assignment One 2025 DES

(1) We have $x(1-2x^2y)y'+y = 3x^2y^2$. $y(1)=1/2$. We put $y = x^{-2}v$.

Then $\frac{dy}{dx} = -\frac{2v}{x^3} + \frac{1}{x^2}\frac{dv}{dx}$. Substituting gives

$$x(1-2v)\left(-\frac{2v}{x^3}+\frac{1}{x^2}\frac{dv}{dx}\right)+\frac{v}{x^2} = \frac{3x^2v^2}{x^4} = \frac{3v^2}{x^2}$$

Which is

$$x^3(1-2v)\left(-\frac{2v}{x^3}+\frac{1}{x^2}\frac{dv}{dx}\right)+v = 3v^2$$

or $(1-2v)\left(-2v+x\frac{dv}{dx}\right)+v = 3v^2$

Giving $(1-2v)x\frac{dv}{dx} = 3v^2-v+2v(1-2v)$
$= -v^2+v = v(1-v)$

or $\frac{1-2v}{v(1-v)}\frac{dv}{dx} = \frac{1}{x}$. Separating gives

$\frac{1-2v}{v(1-v)}dv = \frac{dx}{x}$. The LHS is

$$\left(\frac{1}{v}-\frac{1}{1-v}\right)dv = \frac{dx}{x}$$

Integrating gives $\ln v + \ln(1-v) = \ln x + C$

Hence $v(1-v) = Ax$

Put $v = x^2y$. Then $yx^2(1-yx^2) = Ax$.

Since $y(1)=1/2$

$\frac{1}{2}(1-\frac{1}{2}1^2) = A$ $\therefore$ $A = 1/4$

Giving the implicit solution

$$4yx^2(1-yx^2) = 1$$

(2) Put $y = \frac{h'}{h}$. Then $y' = \frac{h''}{h} - \left(\frac{h'}{h}\right)^2$

Hence $y'+y^2 = \frac{h''}{h} - \left(\frac{h'}{h}\right)^2 + \left(\frac{h'}{h}\right)^2 = \frac{h''}{h} = x$

or $h'' = xh$. So $h = c_1 Ai(x) + c_2 Bi(x)$ and
$y = (c_1 Ai'(x) + c_2 Bi'(x))/(c_1 Ai(x) + c_2 Bi(x))$

(4). First we solve $x^2y''+4xy'-10y=0$

Put $y=x^a$ So $x^2a(a-1)x^{a-2}+4xax^{a-1}-10x^a$

$$=x^a(a^2+3a-10)=0$$

$\therefore$ $a=-5,\ 2$ Take $y_1=x^2,\ y_2=x^{-5}$.

Next $$y''+\frac{4}{x}y'-\frac{10}{x^2}y=x\sin x.$$

Thus $R(x)=x\sin x$, we find the Wronskian

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} x^2 & x^{-5} \\ 2x & -5x^{-6} \end{vmatrix}=-5x^{-4}-2x^{-4}$$
$$=-7x^{-4}$$

Then we find $y_p=uy_1+vy_2$.

$$u'=\frac{-y_2R}{W}=\frac{-x^{-5}\,x\sin x}{-7x^{-4}}=\frac{1}{7}\sin x.$$

$$u=-\frac{1}{7}\cos x.$$

$$v'=\frac{y_1R}{W}=\frac{x^2x\sin x}{-7x^{-4}}=-\frac{1}{7}x^7\sin x$$

Sensible to do this in Mathematica, or Wolfram Alpha

$$v=\frac{x}{7}(x^6-42x^4+840x^2-5040)\cos x$$
$$-7(x^6-30x^4+360x^2-720)\sin x$$

$$y_p=uy_1+vy_2=-\frac{x^2}{7}\cos x+x^{-5}\frac{x}{7}(x^6-42x^4+840x^2-5040)$$
$$x\sin x$$
$$=-\frac{(6x(120-20x^2+x^4)\cos x+(-720+360x^2-30x^4+x^6)\sin x)}{x^5}.$$

General solution is

$$y=c_1x^2+c_2x^{-5}+y_p$$

(4) It is clear that $y=0$ is a solution.
Now we know the solution is analytic about $x=0$. So

$$(*) \quad y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \cdots$$

From the Taylor expansion.

Now $y''(x) = -(p(x)y'(x) + q(x)y(x))$

Thus $y''(0) = -(p(0)y'(0) + q(0)y(0)) = 0$

Because $y(0) = y'(0) = 0$.

But

$$y'''(x) = \frac{d}{dx}y''(x) = -\frac{d}{dx}\left(p(x)y'(x) + q(x)y(x)\right)$$
$$= -\left(p'(x)y'(x) + p(x)y''(x) + q(x)y'(x) + q'(x)y(x)\right)$$

Hence $y'''(0) = -(p'(0)y'(0) + p(0)y''(0) + q(0)y'(0) + q'(0)y(0)) = 0$

as $y''(0)=0$. Since the functions p, q are analytic, $p^{(n)}(0)$, $q^{(n)}(0)$ are finite for all n. Now $y^{(n)}(0)$ depends on $y(0), \ldots, y^{(n-1)}(0)$ and these are always zero.

Hence $y^{(n)}(0) = 0$ all n.

Thus $(*)$ tells us that $y=0$, all $x \in (-a\ a)$

(5) We solve $x^2y'' + x(2x-3)y' + 4y = 0$.

Set $y = \sum_{n=0}^{\infty} a_n x^{n+s}$.

Then $x^2 \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2}$ $x(2x-3)\sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1}$

$+ 4\sum_{n=0}^{\infty} a_n x^{n+s}$

$$= \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s} - \sum_{n=0}^{\infty} 3 a_n (n+s) x^{n+s}$$
$$+ \sum_{n=0}^{\infty} 4 a_n x^{n+s} + \sum_{n=0}^{\infty} 2 a_n x^{n+s+1}$$
$$= a_0 x^s \left(s(s-1) - 3s + 4 \right)$$
$$+ \sum_{n=1}^{\infty} \left((n+s)(n+s-1) - 3(n+s) + 4 \right) a_n x^{n+s}$$
$$+ \sum_{n=0}^{\infty} 2 a_n x^{n+s+1}$$
$$= a_0 x^s (s-2)^2 + \sum_{n=1}^{\infty} \left((n+s)(n+s-1) - 3(n+s) + 4 \right) a_n x^{n+s}$$
$$+ \sum_{n=0}^{\infty} 2 a_n x^{n+s+1} = 0$$

Put $s=2$. Shift second series up to starting value $n=1$.

Then $\displaystyle\sum_{n=1}^{\infty} \left(\left((n+s)(n+s-1) - 3(n+s) + 4 \right) a_n + 2a_{n-1} \right) x^{n+s} = 0$

So $\displaystyle a_n = \frac{-2a_{n-1}}{(n+s)(n+s-1) - 3(n+s) + 4}, \quad n \geq 1$

Put $s=2$.

$$a_n = \frac{-2a_{n-1}}{(n+2)(n+1) - 3(n+2) + 4}$$
$$= \frac{-2a_{n-1}}{(n+2-3)(n+2) + 4}$$
$$= \frac{-2a_{n-1}}{n^2}$$

It follows that $\displaystyle a_n = \frac{(-2)^n a_0}{(n!)^2}$.

So $\displaystyle y = a_0 \sum_{n=0}^{\infty} \frac{(-2)^n x^{n+2}}{(n!)^2}$.

Now $a_n = \dfrac{-2a_{n-1}}{(n+s-2)^2}$. Iterating we have

$$a_n = (-2)^n \left[\frac{a_0}{(s-1)^2(s-2)^2 \cdots (n+s-2)^2} \right]. \text{ Now}$$

$$\ln(a_n) = \ln((-2)^n a_0) - \ln(s-1)^2 \cdots (n+s-2)^2$$

$$= \text{const} - 2\ln((s-1) \cdots (n+s-2))$$

$$\therefore \frac{a_n'(s)}{a_n(s)} = -2\left[\frac{1}{s-1} + \frac{1}{s} + \cdots + \frac{1}{n+s-2}\right]$$

$$\therefore a_n'(2) = -2\left[1 + \frac{1}{2} + \cdots + \frac{1}{n}\right]$$

$$= 2H_n a_n(2)$$

Hence the second solution is

$$y(x) = y_1(x)\ln x + a_0 \sum_{n=1}^{\infty} \frac{-2H_n x^{n+2}}{n^2}.$$