37335 - Differential Equations Assignment

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1 Problem 1

Solve the equation

$$x(1 - 2x^2y)y' + y = 3x^2y^2,$$

$y(1) = \frac{1}{2}$, by setting $y = x^{-2}v$.

1.1 Substitution

Employing the substitution $y = x^{-2}v$ and $y' = x^{-2}v' - 2x^{-3}v$

$$[x - 2x^3(x^{-2}v)](-2x^{-3}v + x^{-2}v') + (x^{-2}v) = 3x^2(x^{-2}v)^2$$
$$(x - 2xv)(-2x^{-3}v + x^{-2}v') + (x^{-2}v) = 3x^{-2}v^2$$
$$(x - 2xv)(-2x^{-1}v + v') + v = 3v^2$$
$$-2v + xv' + 4v^2 - 2xvv' + v = 3v^2$$
$$-v + xv' + 4v^2 - 2xvv' = 3v^2$$
$$xv' - 2xvv' = -v^2 + v$$
$$xv'(1 - 2v) = -v^2 + v$$
$$\frac{v'(1 - 2v)}{-v^2 + v} = \frac{1}{x}$$

Hence the ODE is separable by this neat substitution!

1.1.1 Interesting observation

$x(-2x^2y)y' + y = 3x^2y^2$ can be made made into the exact equations $y' = \frac{3x^2y^2-y}{x-2x^3y}$ or $(2x^3y - x)dy + (3x^2y^2 - y)dx = 0$ One sets $P(x,y) = f_x(x,y) = 3x^2y^2 - y$ and $Q(x,y) = f_y(x,y) = 2x^3y - x$. The potential function is $f(x,y) = x^3y^2 - yx + c$, and hence $x^3y^2 - yx = c$. The quadric formula yields $y(x) = \frac{1 \pm \sqrt{1+4cx}}{2x^2}$, which is another interesting way to solve this ODE. The main derivation of this ODE will be through making the ODE separable however.

1.2 Solving first-order separable ODE

x-integrating both sides of the ODE leads to the following (note the subtle use of the reverse chain rule on the LHS to absorb the derivative factor v' and switch the integral to v-integration)

$$\int \frac{1-2v}{-v^2+v} dv = \int \frac{1}{x} dx$$

Resolving the integral produces the following

$$\ln|v^2 - v| = \ln|x| + c$$
$$v^2 - v = cx$$
$$v^2 - v - cx = 0$$
$$v = \frac{1 \pm \sqrt{1+4cx}}{2}$$

Note that the constant 4 can be absorbed into c.

$$v(x) = \frac{1 \pm \sqrt{1+cx}}{2}$$

$$y(x) = \frac{1 \pm \sqrt{1+cx}}{2x^2}$$

1.3 Applying initial condition

Forcing the general solution to fit the initial condition $y(1) = \frac{1}{2}$ produces the following.

$$\frac{1}{2} = \frac{1 \pm \sqrt{1+c(1)}}{2(1)^2}$$
$$\frac{1}{2} = \frac{1 \pm \sqrt{1+c}}{2}$$
$$1 = 1 \pm \sqrt{1+c}$$
$$0 = \pm\sqrt{1+c}$$
$$c = -1$$

Hence this IVP has the solution below. Note however that the use of the $\pm$ operator in this context means that either $+$ or $-$ can be chosen to obtain a function satisfying the ODE, not in the sense that y has multiple outputs (this violates the definition of a function).

$$y(x) = \frac{1 \pm \sqrt{1-x}}{2x^2}$$

2 Problem 2

Solve the Riccati equation

$$y' + y^2 = x.$$

You will need Airy's equation.

2.1 Mapping to Airy equation

By making the substitution $y(x) = \frac{a(x)u'(x)}{u(x)}$ and $y' = \frac{a'u'}{u} + a\frac{uu''-u'u'}{u^2}$, the differential equation is transformed to the following and can be simplified as demonstrated.

$$[\frac{a'u'}{u} + a\frac{uu'' - u'u'}{u^2}] + \frac{a^2u'u'}{u^2} = x$$

$$\frac{a'u'}{u} + \frac{a}{u^2}[uu'' - u'u' + au'u'] = x$$

Note that the non-linear terms cancel by setting $a(x) = 1$ (hence also setting $a'(x) = 0$). Our substitution is therefore $y(x) = \frac{u'(x)}{u(x)}$.

$$\frac{(0)u'}{u} + \frac{(1)}{u^2}[uu'' - u'u' + (1)u'u'] = x$$

$$\frac{uu''}{u^2} = x$$

$$\frac{u''}{u} = x$$

$$u'' = xu$$

$$u'' - xu = 0$$

$u'' - xu = 0$ is Airy's equation! This may be solved by the series method, however as this is a renown result we omit this tedious process and consider the general solution for u as such.

$$u(x) = c_1\text{Ai}(x) + c_2\text{Bi}(x)$$

2.2 Undoing the substitution

Recalling that the substitution $y(x) = \frac{u'(x)}{u(x)}$ was used to map the ODE to the Airy equation, this intermediate function is now substituted to find the general solution to the original problem.

$$y(x) = \frac{c_1\text{Ai}'(x) + c_2\text{Bi}'(x)}{c_1\text{Ai}(x) + c_2\text{Bi}(x)}$$

3 Problem 3

Use Variation of parameters to solve

$$x^2y'' + 4xy' - 10y = x^3 \sin x.$$

3.1 Solving homogeneous equation by the method of Frobenius

3.1.1 Substituting Frobenius power series

One considers the homogeneous differential equation $x^2y'' + 4xy' - 10y = 0$ and applies the method of Frobenius by substituting $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$.

$$x^2[\sum_{n=0}^{\infty} a_n(n+s)(n+s-1)x^{n+s-2}] + 4x[\sum_{n=0}^{\infty} a_n(n+s)x^{n+s-1}] - 10[\sum_{n=0}^{\infty} a_n x^{n+s}] = 0$$

$$\sum_{n=0}^{\infty} a_n(n+s)(n+s-1)x^{n+s}] + 4[\sum_{n=0}^{\infty} a_n(n+s)x^{n+s}] - 10[\sum_{n=0}^{\infty} a_n x^{n+s}] = 0$$

$$[\sum_{n=0}^{\infty}[a_n(n+s)(n+s-1) + 4a_n(n+s) - 10a_n]x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} a_n[(n+s)(n+s-1) + 4(n+s) - 10]x^{n+s} = 0$$

$$a_n[(n+s)(n+s-1) + 4(n+s) - 10] = 0$$

3.1.2 Calculating s

Since the trivial solution is to be avoided, we assume that $a_0 \neq 0$ and hence one aims to extract the first term out of the series, equate it to 0, and ignores any contibution from a_0.

$$a_0[s(s-1) + 4s - 10]x^{s-2} + \sum_{n=0}^{\infty} a_n[(n+s)(n+s-1) + 4(n+s) - 10] = 0$$

Hence $s(s-1) + 4s - 10 = 0$ and by solving the quadratic it follows that $s = -5, 2$.

$$s^2 - 3s - 10 = 0$$

$$(s+5)(s-2) = 0$$

$$s = -5, 2$$

3.1.3 Calculating (a_n)

Taking $s = -5$ and equating the coefficents to 0 implies the following

$$a_n[(n-5)(n-6)+4(n-5)-10] = 0$$

$$a_n[n^2-7n] = 0$$

Note that the only value of n such that $n^2 - 7 = 0$ is $n = 7$; so we have deduced that $(\forall n \in \mathbb{N} n \neq 7 \implies a_n = 0)$. Considering our Frobenius power series, all the terms of (a_n) zero out except for the 7th term, hence one has $y_1(x) = a_7 x^{7-5} = a_7 x^2$.
Now disregarding the scaling factor a_7 our solution is reduced to $y_1(x) = x^2$.

3.1.4 Calculating linearly independent solution

To avoid laborious expansions of more Frobenius power series, we employ a well-known corollary of the Abel identity to locate a linearly independent solution.

$$y_2(x) = (x^2)\int \frac{e^{-\int(\frac{4}{x})dx}}{(x^2)^2}dx$$

$$= x^2 \int \frac{x^{-4}}{x^4}dx$$

$$= x^2 \int x^{-8}dx$$

$$= x^2(-\frac{x^{-7}}{7})$$

$$= -\frac{1}{7}x^{-5}$$

Hence by disregarding the scaling factor, one reasons that $y_2(x) = x^{-5}$. Now the solutions $y_1(x) = x^2$ and $y_2(x) = x^{-5}$ for the homogeneous equation have been deduced!

$$y_c(x) = c_1 x^2 + c_2 x^{-5}$$

3.2 Variation of Parameters

Thus one now employs VOP to obtain a solution to the inhomogeneous $x^2y'' + 4xy' - 10y = x^3 \sin x$ (we will need this in the standar form $y'' + \frac{4}{x}y' - \frac{10}{x^2}y = x \sin x$), however the second-order linear shortcut of making the ansatz $y(x) = u(x)y_1(x) + v(x)y_2(x)$ leads one to the following set of equations

$$u(x) = -\int \frac{x^{-5}x\sin x}{W(y_1,y_2)}dx$$

$$v(x) = \int \frac{x^{-2}x\sin x}{W(y_1,y_2)}dx$$

, or better yet,

$$u(x) = -\int \frac{x^{-4}\sin x}{W(y_1, y_2)}dx$$
$$v(x) = \int \frac{x^{-1}\sin x}{W(y_1, y_2)}dx$$

One calculates the Wronskian as $W(y_1, y_2) = y_1y_2' - y_2y_1' = (x^2)(-5x^{-6}) - (x^{-5})(2x) = -5x^{-4} - 2x^{-4} = -7x^{-4}$.

$$u(x) = -\int \frac{x^{-5}x\sin x}{-7x^{-4}}dx$$
$$v(x) = \int \frac{x^2 x\sin x}{-7x^{-4}}dx$$

, or better yet,

$$u(x) = \frac{1}{7}\int \sin x dx$$
$$v(x) = -\frac{1}{7}\int x^7 \sin x dx$$

Noting that the sequence of functions $I_n = \int x^n \sin x dx$ follows the following recurrence relation

$$I_n = \begin{cases} -\cos x & n = 0 \\ -x\cos x + \sin x & n = 1 \\ -x^n \cos x + nx^{n-1}\sin x - n(n-1)S_{n-2} n \in \mathbb{N} \cap [2, \infty) \end{cases}$$

By evaluating I_0 and I_7, the following results are derived

$$u(x) = -\frac{\cos x}{7}$$

$$v(x) = \frac{1}{7}(x^7 - 42x^5 + 840x^3 - 5040x)\cos x - (x^6 - 30x^4 + 360x^2 - 720)\sin x$$

Variation of Parameters claims that the particular solution of the ODE has the form

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

Hence by substituting the calculated functions, one has the following function that barely fits on the page.

$$y_p(x) = [-\frac{\cos x}{7}][x^2]+[\frac{1}{7}(x^7-42x^5+840x^3-5040x)\cos x-(x^6-30x^4+360x^2-720)\sin x][x^{-5}]$$

3.3 Linear combination of homogeneous and inhomogeneous solutions

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1x^2+c_2x^{-5}+[-\frac{\cos x}{7}][x^2]+[\frac{1}{7}(x^7-42x^5+840x^3-5040x)\cos x-(x^6-30x^4+360x^2-720)\sin x][x^{-5}]$$

$$y(x) = (c_1-\frac{\cos x}{7})x^2+(c_2+\frac{1}{7}(x^7-42x^5+840x^3-5040x)\cos x-(x^6-30x^4+360x^2-720)\sin x)x^{-5}$$

4 Problem 4

Let p, q be analytic on the interval $I = (-a, a), a > 0$. Show that the IVP

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

, $y(0) = y'(0) = 0$ has $y = 0$ as a solution. Prove that this is the only solution on I.

4.1 $y(x) = 0$ is a solution to the IVP

Define $y(x) = 0$, to prove that it is a solution to the IVP, one must verify that it meets the initial conditions and that it satisfies the ODE.

4.1.1 Checking initial conditions

Since the function 0 and its first derivative are the following.

$$y(x) = 0$$

$$y'(x) = 0$$

it is certainly true that $y(0) = y'(0) = 0$.

4.1.2 Substitution into the ODE

Now what remains is substitution into the differential equation. Noting that the following are 0 and its first two derivatives.

$$y(x) = 0$$
$$y'(x) = 0$$
$$y''(x) = 0$$

substitution into the ODE indeed returns 0.

$$(0) + p(x)(0) + q(x)(0) = 0 + 0 + 0 = 0$$

4.2 Uniqueness of trivial solution on I

Since the coefficients are analytic on I, it is known that solutions of this IVP must be analytic I, so the a power series expanded upon 0 (since $0 \in I$) detects all functions that satisfy the ODE on I. $y(x) = 0$ is the unique solution if this power series sequence is the zero sequence; this will be proven by strong induction. The base cases a_0, a_1 shall be manually calculated and shown to be 0, then under the inductive hypothesis a_n, a_{n-1} are assumed to be 0, and then a_{n+1} must be shown to equal 0 under the inductive hypothesis.

4.2.1 Base cases

The initial conditions of y allow one to see that the first two terms of the Frobenius power series equal 0.
When evaluating $f(x) = \sum_{n=0}^{\infty} a_n x^n$ at 0, one has $f(0) = a_0$. This is because $\lim_{x\to 0} x^0 = 1$ and $n > 0 \implies \lim_{x\to 0} x^n = 0$. Similarly, $g(x) = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n$ and by the same reasoning $g(0) = (0+1)a_{0+1} = a_1$.

$$y(0) = 0 \implies \sum_{n=0}^{\infty} a_n (0)^n = 0 \implies a_0 = 0$$

$$y'(0) = 0 \implies \sum_{n=1}^{\infty} a_n n (0)^{n-1} = 0 \implies a_1 = 0$$

4.2.2 Hypothesis cases

By the hypothesis of strong induction with two base cases, the subsequent argument follows

$$a_n = a_{n-1} = 0$$

Now there remains to show that $\forall n \geq 0 (a_n = 0)$;

4.2.3 Inductive case

Now it must be proven that $a_{n+1} = 0$. This can be done by leveraging the inductive hypothesis and the closed form for the power series sequence as such.

$$[\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}] + p(x)[\sum_{n=1}^{\infty} a_n n x^{n-1}] + q(x)[\sum_{n=0}^{\infty} a_n x^n] = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} a_{n-1}(n-1)p(x)x^{n-2} + \sum_{n=2}^{\infty} a_{n-2} q(x) x^{n-2} = 0$$

$$\sum_{n=2}^{\infty} [n(n-1)a_n + a_{n-1}(n-1)p(x) + a_{n-2}q(x)]x^{n-2} = 0$$

$$n(n-1)a_n + a_{n-1}(n-1)p(x) + a_{n-2}q(x) = 0$$

$$a_n = -\frac{a_{n-1}(n-1)p(x) + a_{n-2}q(x)}{n(n-1)}$$

Now that a recursive form for (a_n) is known, sufficient background is achieved to finish the inductive proof.

$$a_n = -\frac{a_{n-1}(n-1)p(x) + a_{n-2}q(x)}{n(n-1)}$$

$$a_{n+1} = -\frac{a_n np(x) + a_{n-1}q(x)}{n(n+1)}$$

$$a_{n+1} = -\frac{(0)np(x) + (0)q(x)}{n(n+1)}$$

$$a_{n+1} = 0$$

Therefore by strong induction, we have proven that $\forall n \geq 0(a_n = 0)$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (0)x^n = 0$$

5 Problem 5

Solve the ODE

$$x^2 y'' + x(2x-3)y' + 4y = 0,$$

by the method of Frobenius.

5.1 Substituting Frobenius power series

Consider the ODE in its standard form

$$y'' + \frac{2x-3}{x}y' + \frac{4}{x^2}y = 0$$

. Since there exists a singularity at $x = 0$, one employs the Frobenius method by making the ansatz $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$.

$$x^2[\sum_{n=0}^{\infty} a_n(n+s)(n+s-1)x^{n+s-2}] + x(2x-3)[\sum_{n=0}^{\infty} a_n(n+s)x^{n+s-1}] + 4[\sum_{n=0}^{\infty} a_n x^{n+s}] = 0$$

$$\sum_{n=0}^{\infty} a_n(n+s)(n+s-1)x^{n+s} + (2x-3)\sum_{n=0}^{\infty} a_n(n+s)x^{n+s} + 4\sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} a_n[(n+s)(n+s-1)+4]x^{n+s} + (2x-3)\sum_{n=0}^{\infty} a_n(n+s)x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} a_n[(n+s)(n+s-1)+4-3(n+s)]x^{n+s} + 2x\sum_{n=0}^{\infty} a_n(n+s)x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} a_n[(n+s)(n+s-4)+4]x^{n+s} + 2\sum_{n=0}^{\infty} a_n(n+s)x^{n+s+1} = 0$$

$$\sum_{n=0}^{\infty} a_n[(n+s)(n+s-4)+4]x^{n+s} + \sum_{n=1}^{\infty} 2a_{n-1}(n+s-1)x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} a_n[(n+s)(n+s-4)+4]x^{n+s} + \sum_{n=1}^{\infty} 2a_{n-1}(n+s-1)x^{n+s} = 0$$

$$a_0[s(s-4)+4]x^s + \sum_{n=1}^{\infty} a_n[(n+s)(n+s-4)+4]x^{n+s} + \sum_{n=1}^{\infty} 2a_{n-1}(n+s-1)x^{n+s} = 0$$

$$a_0[s(s-4)+4]x^s + \sum_{n=1}^{\infty}[a_n[(n+s)(n+s-4)+4] + 2a_{n-1}(n+s-1)]x^{n+s} = 0$$

As a preliminary note, by equating the coefficients to 0 one has the following $\forall n \geq 1$.

$$a_n[4+(n+s)(n+s-4)] + 2a_{n-1}(n+s-1) = 0$$

$$a_n = -\frac{2a_{n-1}(n+s-1)}{4+(n+s)(n+s-4)}$$

This will prove useful in future when deriving a second linearly independent solution.

5.2 Calculating s

Now one equates the coefficiets to 0 to solve for s; we look at the first term. Since we disallow $a_0 = 0$ to avoid the trivial solution, Solving $4 + s(s-4) = 0$ implies that $s = 2$ (a root of double multiplicity, which will come to haunt us later).

$$a_0[s(s-4)+4] = 0$$

$$s^2 - 4s + 4 = 0$$

$$(s-2)^2] = 0$$

$$s = 2$$

5.3 Calculating (a_n)

Recall the preliminary note that holds $\forall n \geq 1$.

$$a_n = -\frac{2a_{n-1}(n+s-1)}{4+(n+s)(n+s-4)}$$

Making the substitution $s = 2$ leads to the following.

$$a_n = -\frac{2a_{n-1}(n+1)}{4+(n+2)(n-2)}$$

$$a_n = -\frac{2a_{n-1}(n+1)}{n^2}$$

Then solving for closed form produces the following result.

$$a_n = a_0 \frac{(-2)^n(n+1)}{n!}$$

Hence by setting $s = 2$, subsituting the power series sequence with its closed form and dividing out the scaling constant a_0, one has

$$y_1(x) = x^2 + x^2 \sum_{n=1}^{\infty} \frac{(-2)^n(n+1)}{n!} x^n$$

Considerable simplification of this function can be made to reduce it to an elementary form!

$$y_1(x) = x^2 + x^2 \sum_{n=1}^{\infty} \frac{(-2)^n(n+1)}{n!} x^n$$

$$y_1(x) = x^2 + x^2 [\sum_{n=1}^{\infty} \frac{(-2x)^n}{n!} + \sum_{n=1}^{\infty} \frac{n(-2x)^n}{n!}]$$

$$y_1(x) = x^2 + x^2 [(e^{-2x} - 1) + \sum_{n=1}^{\infty} \frac{n(-2x)^n}{n!}]$$

$$y_1(x) = x^2 + x^2 [(e^{-2x} - 1) + \sum_{n=1}^{\infty} \frac{(-2x)^n}{(n-1)!}]$$

$$y_1(x) = x^2 + x^2 [(e^{-2x} - 1) - 2x \sum_{n=1}^{\infty} \frac{(-2x)^{n-1}}{(n-1)!}]$$

$$y_1(x) = x^2 + x^2 [(e^{-2x} - 1) - 2x(e^{-2x})]$$

$$y_1(x) = x^2 + x^2 [e^{-2x} - 1 - 2xe^{-2x}]$$

$$y_1(x) = x^2 (e^{-2x} - 2xe^{-2x})$$

$$y_1(x) = x^2 e^{-2x} (1 - 2x)$$

5.4 Calculating linearly independent Frobenius power series

To find a second linearly independent solution in the case of only one s, the following formula is used.

$$y_2(x) = y_1(x)\ln(x) + x^s \sum_{n=1}^{\infty} a_n'(s)x^n$$

5.4.1 s-Differentiating a_n

It will be necessary to s-differentiate a_n, hence we shall recall the preliminary result from earlier as a starting point.

$$a_n(s) = -\frac{2a_{n-1}(n+s-1)}{4+(n+s)(n+s-4)}$$

$$a_n(s) = \frac{(-2)^n \prod_{k=1}^{n}(k+s-1)}{\prod_{k=1}^{n}[4+(k+s)(k+s-4)]}a_0$$

$$\ln|a_n(s)| = \ln|(-2)^n a_0 \prod_{k=1}^{n}(k+s-1)| - \ln|\prod_{k=1}^{n}[4+(k+s)(k+s-4)]|$$

$$\ln|a_n(s)| = \ln|a_0| + n\ln|-2| + \sum_{k=1}^{n}\ln|k+s-1| - \sum_{k=1}^{n}\ln|4+(k+s)(k+s-4)|$$

$$\frac{a_n'(s)}{a_n(s)} = \sum_{k=1}^{n}\frac{1}{k+s-1} - \sum_{k=1}^{n}\frac{2s+2k-4}{4+(k+s)(k+s-4)}$$

$$a_n'(s) = (\frac{(-2)^n \prod_{k=1}^{n}(k+s-1)}{\prod_{k=1}^{n}[4+(k+s)(k+s-4)]}a_0)(\sum_{k=1}^{n}\frac{1}{k+s-1} - \sum_{k=1}^{n}\frac{2s+2k-4}{4+(k+s)(k+s-4)})$$

Now this function is evaluated at the double root 2.

$$a_n'(2) = (\frac{(-2)^n(n+1)!}{(n!)^2}a_0)(\sum_{k=1}^{n}\frac{1}{k+1} - \sum_{k=1}^{n}\frac{2k}{4+(k+2)(k-2)})$$

$$a_n'(2) = (\frac{(-2)^n(n+1)!}{(n!)^2}a_0)(\sum_{k=1}^{n}\frac{1}{k+1} - 2\sum_{k=1}^{n}\frac{k}{k^2})$$

$$a_n'(2) = (\frac{(-2)^n(n+1)}{n!}a_0)(\sum_{k=2}^{n+1}\frac{1}{k} - 2\sum_{k=1}^{n}\frac{1}{k})$$

$$a_n'(2) = a_0\frac{(-2)^n(n+1)(H_{n+1}-1-2H_n)}{n!}$$

5.4.2 Applying the formula

A simple application of the aforementioned formula produces the desired result.

$$y_2(x) = [x^2 e^{-2x}(1-2x)]\ln(x) + a_0 \sum_{n=1}^{\infty} \frac{(-2)^n(n+1)(H_n - 1 - 2H_n)}{n!} x^{n+2}$$

5.5 Linear combination of linearly independent solutions

Indeed, $y(x) = c_1 y_1(x) + c_2 y_2(x)$, so therefore the following is the general solution.

$$y(x) = c_1[x^2 e^{-2x}(1-2x)] + c_2[[x^2 e^{-2x}(1-2x)]\ln(x) + a_0 \sum_{n=1}^{\infty} \frac{(-2)^n(n+1)(H_n - 1 - 2H_n)}{n!} x^{n+2}]$$