Differential Equations

Autumn 2025 Assignment Two. Due the final tutorial.

(1) The Laplace transform can be used to solve a wide range of problems. Use the Laplace transform to solve the equation

$$y'(t) - 2\int_0^t y(u)\cos(t-u)du = 1,$$

with $y(0) = -1$. Hint: Convolution.

(2) It is possible to expand periodic functions in terms of functions other than sines and cosines. A differentiable function f on $(0,1)$ can be written as

$$f(x) = \sum_{k=1}^{\infty} A_k J_n(\lambda_k x),\ x \in (0,1),$$

where J_n are order n Bessel functions and $\lambda_1, \lambda_2, \lambda_3....$ are the zeroes of J_n. That is $J_n(\lambda_k) = 0, k = 1, 2, 3....$ The problem is to find A_k. We have two useful facts which you can assume

$$\int_0^1 xJ_n(ax)J_n(bx)dx = \frac{bJ_n(a)J_n'(b) - aJ_n(b)J_n'(a)}{a^2 - b^2},\ a \neq b,$$

$$\int_0^1 xJ_n^2(ax)dx = \frac{1}{2}\left[(J_n'(a))^2 + \left(1 - \frac{n^2}{a^2}\right)(J_n(a))^2\right].$$

Look carefully at how the formula for the Fourier coefficients are derived. Mimic this procedure to show that

$$A_k = \frac{2}{(J_{n+1}(\lambda_k))^2}\int_0^1 xf(x)J_n(\lambda_k x)dx.$$

(3) This uses question 2 to solve an important engineering problem. We have a circular plate, such as a stove top. Take its radius to be 1. The temperature of the plate depends only on the distance r from the origin. The initial temperature is $f(r)$ and the temperature at $r = 1$ is kept equal to zero. The temperature u at time t will satisfy the partial differential equation

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r},$$
$$u(1,t) = 0, u(r,0) = f(r), |u(r,t)| < M,$$

for some finite $M > 0$. This last condition just says that the temperature is finite. k is a constant depending on the plate. Use separation of variables to solve this problem. That is, let $u(r,t) = R(r)T(t)$ and follow the procedure we used in lectures for the heat equation. You will need the fact that $|Y_0(x)| \to \infty$ as $x \to 0^+$.