37335 - Differential Equations Assignment 2

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1 Problem 1

The Laplace transform can be used to solve a wide range of problems. Use the Laplace transform to solve the equation

$$y'(t) - 2\int_0^t y(u)\cos(t-u)du = 1,$$

with $y(0) = -1$. Hint : Convolution.

1.1 Laplace transform

$$\mathcal{L}\{y'(t) - 2\int_0^t y(u)\cos(t-u)du\}(s) = \mathcal{L}\{1\}(s),$$

$$\mathcal{L}\{y'\}(s) - 2\mathcal{L}\{\int_0^t y(u)\cos(t-u)du\}(s) = \mathcal{L}\{1\}(s)$$

$$\mathcal{L}\{y'\}(s) - 2\mathcal{L}\{\int_0^t y(u)\cos(t-u)du\}(s) = \frac{1}{s}$$

$$\mathcal{L}\{y'\}(s) - 2\mathcal{L}\{y * \cos t\}(s) = \frac{1}{s}$$

1.1.1 Convolution theorem

$$\mathcal{L}\{y'\}(s) - 2\mathcal{L}\{y\}(s)\mathcal{L}\{\cos t\}(s) = \frac{1}{s}$$

$$\mathcal{L}\{y'\}(s) - 2\mathcal{L}\{y\}(s)\frac{1}{s^2+1} = \frac{1}{s}$$

1.1.2 Differentiation proposition

$$s\mathcal{L}\{y\}(s) - y(0) - 2\mathcal{L}\{y\}(s)\frac{1}{s^2+1} = \frac{1}{s}$$

$$s\mathcal{L}\{y\}(s) + 1 - 2\mathcal{L}\{y\}(s)\frac{1}{s^2+1} = \frac{1}{s}$$

$$su + 1 - \frac{2su}{s^2+1} = \frac{1}{s}$$

$$u(s - \frac{2s}{s^2+1}) = \frac{1-s}{s}$$

$$u(s - \frac{2s}{s^2+1}) = \frac{-(s-1)}{s}$$

$$u(s(s^2+1) - 2s) = \frac{-(s-1)(s^2+1)}{s}$$

$$u(s^3 - s) = \frac{-(s-1)(s^2+1)}{s}$$

$$us(s-1)(s+1) = \frac{-(s-1)(s^2+1)}{s}$$

$$u = \frac{-(s^2+1)}{s^2(s+1)}$$

1.2 Inverse Laplace transform

The natural way to begin an inverse Laplace transform is to exploit any linearity of terms as such.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{\frac{-(s^2+1)}{s^2(s+1)}\}(t) \\
&= \mathcal{L}^{-1}\{\frac{-(s^2+1)}{s^2(s+1)}\}(t) \\
&= \mathcal{L}^{-1}\{\frac{-s^2-1)}{s^2(s+1)}\}(t) \\
&= \mathcal{L}^{-1}\{\frac{-s^2}{s^2(s+1)} + \frac{-1}{s^2(s+1)}\}(t) \\
&= -\mathcal{L}^{-1}\{\frac{s^2}{s^2(s+1)}\}(t) - \mathcal{L}^{-1}\{\frac{1}{s^2(s+1)}\}(t) \\
&= -\mathcal{L}^{-1}\{\frac{1}{s+1}\}(t) - \mathcal{L}^{-1}\{\frac{1}{s^2(s+1)}\}(t)
\end{aligned}$$

By reversing the Laplace transform $\mathcal{L}\{\frac{1}{s+n}\} = e^{-nt}$ one deduces the following.

$$= -e^{-t} - \mathcal{L}^{-1}\{\frac{1}{s^2(s+1)}\}(t)$$

By reversing the convolution theorem one inverses the product of two Laplace transforms as the convolution of their respective inverses. Further noting that $\mathcal{L}\{\frac{\Gamma(n+1)}{s^{n+1}}\} = t^n$, one is lead to the following.

$$= -e^{-t} - (e^{-t} * t)(t)$$

The convolution can be represented in terms of elementary functions by the reverse product rule.

$$\begin{aligned}
(e^{-t} * t)(t) &= \int_0^t (t-u)e^{-u}du \\
&= [-e^{-u}(t-u)]_0^t - \int_0^t -e^{-u}du \\
&= t - \int_0^t -e^{-u}du \\
&= t - [e^{-u}]_0^t \\
&= t - (e^{-t} + 1) \\
&= t - e^{-t} - 1
\end{aligned}$$

Substituting this into the original expression gives the following.

$$\begin{aligned}
y(t) &= -e^{-t} - (e^{-t} + t - 1) \\
&= 1 - 2e^{-t} - t
\end{aligned}$$

Hence the ODE is satisfied by the following function.

$$y(t) = 1 - 2e^{-t} - t$$

2 Problem 2

It is possible to expand periodic functions in temrs of functions other than sines and cosines. A differentiable function f on $(0,1)$ can be written as

$$f(x) = \sum_{k=1}^{\infty} A_k J_n(\lambda_k x), x \in (0,1),$$

where J_n are order n Bessel functions and $\{\lambda_k : k \in \mathbb{N} \setminus \{0\}\}$ is the set of zeroes of J_n. That is $J_n(\lambda_k) = 0, k \in \mathbb{N} \setminus \{0\}$. The problem is to find A_k. We have two useful facts which you can assume

Look carefully at how the formula for the Fourier coefficients are derived. Mimic this procedure to show that

$$A_k = \frac{2}{(J_{n+1}(\lambda_k))^2} \int_0^1 x f(x) J_n(\lambda_k x) dx$$

2.1 Taking x-weighted inner product

As the problem suggests, the set $\{J_n(\lambda_k x) : k \in \mathbb{N} \setminus \{1\}\}$ forms the basis for the function space $L^1(0,1)$, however interestingly this function space uses a weighted inner product, with weight x. In a similar fashion to the Fourier series, one equates the inner product of both sides of the equation, taken with the second argument being an arbitrary basis element $J_n(\lambda_m x)$.

$$f(x) = \sum_{k=1}^{\infty} A_k J_n(\lambda_k x)$$

$$xf(x)J_n(\lambda_m x) = \sum_{k=1}^{\infty} A_k J_n(\lambda_k x)J_n(\lambda_m x)x$$

$$\int_0^1 xf(x)J_n(\lambda_m x)dx = \int_0^1 \sum_{k=1}^{\infty} A_k J_n(\lambda_k x)J_n(\lambda_m x)xdx$$

2.2 Evaluating integrals

Ideally one would like to pass the integral though the limit. The dominated convergence theorem (DCT) ensures the ability to swap limits if on $(0,1)$ there is an $L^1(0,1)$ function g (note also that f should be in $L^1(0,1)$ if we are integrating over it at all, which we are) that dominates each sequence term (in this case, the partial sums). Since such a function can be found, limit swapping is justified.

$$\int_0^1 xf(x)J_n(\lambda_m x)dx = \sum_{k=1}^{\infty} A_k \int_0^1 J_n(\lambda_k x)J_n(\lambda_m x)xdx$$

$$\int_0^1 xf(x)J_n(\lambda_m x)dx = \frac{A_m}{2}[(J_n'(\lambda_m))^2 + (1 - \frac{n^2}{\lambda_m^2})(J_n(\lambda_m))^2] + \sum_{k \in \mathbb{N} \setminus \{m,0\}} A_k \frac{\lambda_k J_n(\lambda_m)J_n'(\lambda_k) - \lambda_m J_n(\lambda_k)J_n'(\lambda_m)}{\lambda_m^2 - \lambda_k^2}$$

Noting that $\forall k \in \mathbb{N} \setminus \{0\}[J_n(\lambda_k) = 0]$, one has the following.

$$\int_0^1 xf(x)J_n(\lambda_m x)dx = \frac{A_m}{2}[(J_n'(\lambda_m))^2]$$

$$A_m = \frac{2}{(J_n'(\lambda_m))^2} \int_0^1 xf(x)J_n(\lambda_m x)dx$$

2.3 Expressing J_n' as a Bessel function

A lemma relating n order Bessel function derivatives to Bessel functions of other orders will now be proven. Note the following manipulation of the n order Bessel function.

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!}(\frac{x}{2})^{2m+n}$$

$$J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m(2m+n)x^{2m+n-1}}{m!(m+n)!2^{2m+n}}$$

$$J_n'(x) = \sum_{m=0}^{\infty} 2m\frac{(-1)^m x^{2m+n-1}}{m!(m+n)!2^{2m+n}} + n\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n-1}}{m!(m+n)!2^{2m+n}}$$

$$J_n'(x) = \sum_{m=0}^{\infty} 2m\frac{(-1)^m x^{2m+n-1}}{m!(m+n)!2^{2m+n}} + \frac{n}{x}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!}(\frac{x}{2})^{2m+n}$$

$$J_n'(x) = \sum_{m=0}^{\infty} 2m\frac{(-1)^m x^{2m+n-1}}{m!(m+n)!2^{2m+n}} + \frac{n}{x}J_n(x)$$

$$J_n'(x) = \sum_{m=-1}^{\infty} 2(m+1)\frac{(-1)^{m+1} x^{2(m+1)+n-1}}{(m+1)!(m+n+1)!2^{2(m+1)+n}} + \frac{n}{x}J_n(x)$$

$$J_n'(x) = \sum_{m=0}^{\infty} 2(m+1)\frac{(-1)^{m+1} x^{2(m+1)+n-1}}{(m+1)!(m+n+1)!2^{2(m+1)+n}} + \frac{n}{x}J_n(x)$$

$$J_n'(x) = -\sum_{m=0}^{\infty} \frac{(-1)^m x^{2(m+1)+n-1}}{m!(m+n+1)!2^{2(m+1)+n-1}} + \frac{n}{x}J_n(x)$$

$$J_n'(x) = -\sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n+1)!}(\frac{x}{2})^{2m+n+1} + \frac{n}{x}J_n(x)$$

$$J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$$

$$J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$$

Recalling that $J_n(\lambda_m) = 0$ and applying this lemma completes the proof.

$$A_m = \frac{2}{(\frac{n}{x}J_n(\lambda_m) - J_{n+1}(\lambda_m))^2}\int_0^1 xf(x)J_n(\lambda_m x)dx$$

$$A_m = \frac{2}{(-J_{n+1}(\lambda_m))^2}\int_0^1 xf(x)J_n(\lambda_m x)dx$$

$$A_m = \frac{2}{(J_{n+1}(\lambda_m))^2}\int_0^1 xf(x)J_n(\lambda_m x)dx$$

3 Problem 3

This uses question 2 to solve an important engineering problem. We have a circular playe, such as a stove top. Take its radius to be 1. The temperature of the plate depends only on the distance r from the origin. The initial temperature is $f(r)$ and the temperature at $r = 1$ is kept equal to zero. The temperature u at time t will satisfy the partial differential equation

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r},$$
$$u(1,t) = 0, u(r,0) = f(r), |u(r,t)| < M,$$

for some finite $M > 0$. This last condition just says that the temperature is finite. k is a constant depending on the plate. Use separatin of variables to solve this problem. That is, let $u(r,t) = R(r)T(t)$ and follow the procedure we used in the lectures for the heat equation. You will need the fact that $\lim_{x\to 0^+} |Y_0(x)| = \infty$.

3.1 Applying separation of variables

One makes the ansatz $u(r,t) = R(r)T(t)$

$$\frac{1}{k}RT' = R''T + \frac{1}{r}R'T$$
$$\frac{1}{k}RT' = T(R'' + \frac{1}{r}R')$$
$$\frac{T'}{T} = \frac{kR'' + \frac{k}{r}R'}{R}$$

Since this relation holds for any setting of r, t, this implies both sides of the equation equate to a constant. Let such a constant be denoted as μ.

$$\frac{T'}{T} = \mu, \frac{kR'' + \frac{k}{r}R'}{R} = \mu$$

Now one seeks to solve these two resultant ODEs.

$$T' = \mu T$$

This is a first order linear ODE with the following general solution.

$$T(t) = c_t e^{\mu t}$$

One now turns to the other ODE.

$$r^2R'' + rR' - r^2\frac{\mu}{k}R = 0$$

This second order linear ODE can be mapped to a Bessel equation! The following lemma is used to map this ODE to a Bessel equation.

$$x^2y'' + (1-2s)xy' + [(s^2 - r^2\alpha^2) + a^2r^2x^2]y = 0$$
$$\implies y(x) = c_1x^sJ_\alpha(ax^r) + c_2x^sY_\alpha(ax^r)$$

Letting $\mu < 0$ and considering that the solution must be bounded on $[0,1]$, one must omit the Bessel function of the second kind from the solution, hence one has the following.

$$R(r) = c_rJ_0(\sqrt{\frac{-\mu}{k}}r)$$

The Dirichlet condition $u(1,t) = 0$ allows the deduction of μ.

$$0 = c_rJ_0(\sqrt{\frac{-\mu}{k}})$$

One avoids the possibility that $c_r = 0$ since this leads to the trivial solution, instead one considers $0 = J_0(\sqrt{\frac{-\mu}{k}})$, hence the argument within J_0 must be a zero of J_0; this article denotes the sequence of ascending real zeroes of J_0 as $(\lambda_n)_{n\in\mathbb{N}\setminus\{0\}}$. This implies the following.

$$\sqrt{\frac{-\mu}{k}} = \lambda_n$$

$$\mu = -k\lambda_n^2$$

Considering the linearity of the PDE, infinite sets of solutions, and condensing the coefficients of R, T into one c_n, the solution now turns toward the following Bessel-Fourier expansion.

$$u(r,t) = \sum_{n=1}^{\infty} c_ne^{-k\lambda_n^2}J_0(\lambda_nr)$$

3.2 Calculating Bessel-Fourier coefficients

Now one considers the condition $u(r,0) = f(r)$. Applying this to the unrefined solution leads to the following.

$$f(r) = \sum_{n=1}^{\infty} c_nJ_0(\lambda_nr)$$

By problem 2, the Bessel-Fourier coefficients for a $L^1([0,1])$ function is the following.

$$c_n = \frac{2}{(J_1(\lambda_n))^2}\int_0^1 rf(r)J_0(\lambda_nr)dr$$

Therefore given $k, f(r)$, one has the following solution to the PDE.

$$u(r,t) = \sum_{n=1}^{\infty} \frac{2}{(J_1(\lambda_n))^2}e^{-k\lambda_n^2}J_0(\lambda_nr)\int_0^1 rf(r)J_0(\lambda_nr)dr$$