UNIVERSITY OF TECHNOLOGY, SYDNEY
SCHOOL OF MATHEMATICAL SCIENCES
35231 Differential Equations, Spring 2010
Class Test. Time Allowed: 55 minutes

(1) (a) Given that $y_1(x) = x$ is a solution of the DE

$$x^3y'' - 2xy' + 2y = 0,$$

find a second linearly independent solution.

(b) Solve the equation $y'' + y = \tan x$ by variation of parameters. You may use the fact that $\sec x = \frac{\cos x}{1-\sin^2 x}$.

(2) Obtain the general solution of the equation

$$y'' - 2xy' + 4y = 0,$$

using a power series expansion of the form $y = \sum_{n=0}^{\infty} a_n x^n$.

(3) The Laplace transform of a suitable function $f(t)$ is given by $F(s) = \int_0^\infty e^{-st} f(t)dt$.

(a) Given that the Laplace transform of f is $F(s)$, obtain the Laplace transform of $\int_0^t f(u)du$.

(b) Solve the ODE

$$y'' + 5y' + 6y = 1, \; y(0) = 0, \; y'(0) = 1,$$

using Laplace transform.

You may need the following information.

If $\mathcal{L}$ denotes the Laplace transform, then

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}.$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}.$$

$$\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}.$$

$$\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}.$$

Marking scheme (1)(i) 6 marks: 2 for $y_2 = y_1 \int \frac{e^{\int p(x)}}{y_1^2} dx$
1 for $p(x)$. 3 for working and final answer. } 6

(ii) 2 for y_1, y_2. 1 for $W(y_1, y_2)$
2 for u', 2 for v'
1 for u, 3 for v, 2 for answer. } 13

(2) 2 for $a_2 = -2a_0$

3 for $a_{n+2} = \frac{2(n-2)}{(n+2)(n+1)} a_n$

2 for $a_{n+2} = 0 \quad n \geq 2$

4 for odd terms
3 for final solution
accept answer in form
$$y = a_0(1-2x^2) + a_1\left(x - \frac{1}{3}x^3 \mp \frac{4.3}{5!}x^5 - \dots\right)$$
as long as coefficients are correct } 14

(3) (i) ~~(3) 1 for con~~ 2 for convolution thm
2 for answer } 4

(ii) 2 for $\mathcal{L}(y'')$, 1 for $\mathcal{L}(y')$

2 for $(s^2+5s+6)Y(s) = \frac{1}{s} + 1$

2 for $\frac{1}{s^2+5s+6} = \frac{1}{s+2} - \frac{1}{s+3}$

3 for $\frac{1}{s(s^2+5s+6)} = \frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}$

3 for answer } 13

(1) (i) $y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = 0$ has $y = x$ as a solution

A second solution is

$$y_2 = y_1 \int \frac{1}{x^2} e^{-\int p(x)dx} dx \qquad p(x) = -\frac{2}{x^2}$$

So $\int p(x)dx = \frac{2}{x}$

Hence $y_2 = x \int \frac{1}{x^2} e^{-\frac{2}{x}} dx$ put $x = \frac{1}{u}$

then $du = \frac{-1}{x^2}dx$ so $\int \frac{1}{x^2} e^{-\frac{2}{x}} dx = -\int e^{-2u} du$

$$= \frac{1}{2} e^{-2u}.$$

Thus $y_2 = \frac{x}{2} e^{-\frac{2}{x}}$ is a second solution

(ii) $y'' + y = \tan x$. $y'' + y = 0$ has solutions $y_1 = \sin x$ and $y_2 = \cos x$. $W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$

So we look for $y_p = uy_1 + vy_2$

$$u' = -\frac{y_2 R}{W} = \frac{-\cos x \tan x}{-1} = \sin x.$$

$$\therefore \quad u = -\cos x.$$

$$v' = \frac{y_1 R}{W} = -\sin x \tan x = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x}$$

$$= \cos x - \sec x$$

$$\sec x = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x}$$

So $\int \sec x\, dx = \int \frac{\cos x}{1-\sin^2 x} dx = \int \frac{dz}{1-z^2} \qquad z = \sin x$

$$= \int \frac{1}{2}\left[\frac{1}{1+z} + \frac{1}{1-z}\right] dz = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right)$$

Hence $v = \int (\cos x - \sec x)\,dx = \sin x - \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right)$

and $y_p = uy_1 + vy_2 = \cancel{-\cos x\sin x} + \cancel{\cos x \sin x} - \frac{1}{2}\cos x \ln\left(\frac{1+\sin x}{1-\sin x}\right)$

$$= -\frac{1}{2}\cos x \ln\left(\frac{1+\sin x}{1-\sin x}\right).$$

General solution is $y = C_1\cos x + C_2 \sin x - \frac{1}{2}\cos x \ln\left(\frac{1+\sin x}{1-\sin x}\right)$.

(2) Let $y = \sum_{n=0}^{\infty} a_n x^n$ Then

$y'' - 2xy' + 4y = 0$ implies

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

or

$$2a_2 + 4a_0 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=1}^{\infty} 4a_n x^n$$

$$= 2a_2 + 4a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - (2n-4)a_n\right]x^n = 0$$

Hence $a_2 = -2a_0$, $a_{n+2} = \dfrac{2(n-2)}{(n+2)(n+1)}a_n$

Now

$a_4 = 2(2-2)a_2 = 0.$, $a_6 = \dfrac{2(4-2)}{(6)(5)}a_4 = 0$

etc. So $a_2 = -2a_0$, $\boxed{a_4 = a_6 = a_8 = \dots = 0}$.

For the odd terms

$$a_3 = \frac{2(1-2)}{3.2}a_1 = \frac{-2}{3.2}a_1$$

$$a_5 = \frac{2(3-2)}{5.4}a_3 = \frac{-2^2}{5!}a_1,\quad a_7 = \frac{2(5-2)}{7.6}a_5$$

$$= \frac{-2^3\, 1.1.3}{7!}a_1,\quad a_9 = \frac{-2^4\, 1.1.3.5}{9!}a_1$$

etc. $a_{2n+1} = -\dfrac{2^n\, 1.1.3.\,(2n-3)}{(2n+1)!}a_1$

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Thus $y = a_0(1-2x^2) + a_1\left(x - \frac{1}{3}x^3 - \sum_{n=2}^{\infty} \frac{2^n 1.3\ldots(2n-3)}{(2n+1)!} x^{2n+1}\right)$

is the general solution. or equiv.

3) (a) $\mathcal{L}(f) = F$. $\int_0^t f(u)du = (f * 1)$. (for convolution)

So $\mathcal{L}\left(\int_0^t f(u)du\right) = F(s)\mathcal{L}(1) = \frac{1}{s}F(s)$

b) $y'' + 5y' + 6y = 1$, $y(0) = 0$, $y'(0) = 1$. $Y(s) = \mathcal{L}(y)$

$s^2 Y(s) - sy(0) - y'(0) = \mathcal{L}(y'')$, $\mathcal{L}(1) = \frac{1}{s}$

$sY(s) - y(0) = \mathcal{L}(y')$,

So $(s^2 + 5s + 6)Y(s) - 1 = \frac{1}{s}$

or $Y(s) = \frac{1}{s^2+5s+6} + \frac{1}{s(s^2+5s+6)}$

$$\frac{1}{s^2+5s+6} = \frac{1}{(s+3)(s+2)} = \frac{1}{s+2} - \frac{1}{s+3}$$

$$\frac{1}{s(s+2)(s+3)} = \frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}.$$

or $Y(s) = \frac{1}{6s} + \frac{1}{2(s+2)} - \frac{2}{3(s+3)}$.

Hence

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{1}{6} + \frac{1}{2}e^{-2t} - \frac{2}{3}e^{-3t}.$$