Question 1 (5 + 10 + 5 = 20 marks).

(a) The ordinary differential equation

$$xy'' + (1 - 2x)y' + (x - 1)y = 0,$$

has a solution $y_1 = e^x$. Use this to find a second linearly independent solution y_2.

(b) Use Variation of Parameters to solve

$$x^2y'' - 3xy' + 3y = 2x^4e^x.$$

The homogenous problem has solutions of the form $y = x^r$.

(c) Express the solution of the equation

$$x^2y'' + 5xy' + (4 - x^6)y = 0$$

in terms of Bessel functions.

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Question 2 (10 + 10 = 20 marks).

(a) Use a power series to obtain the general solution of the ODE

$$(x^2 - 1)y'' + 2xy' - 2y = 0.$$

(b) Consider the ODE

$$2x^2y'' + 3xy' - (1 + x)y = 0.$$

(i) Look for a solution of the form $y = x^s \sum_{n=0}^{\infty} a_n x^n$. Show that we must have $s = -1$ or $s = 1/2$.

(ii) Show that the coefficients satisfy

$$a_{n+1} = \frac{a_n}{(n + s + 2)(2n + 2s + 1)}.$$

(iii) Use the recurrence relation for a_n to generate two linearly independent solutions for the ODE.

Question 3 (5 + 10 + 3 + 2 = 20 marks).

(a) Calculate the inverse Laplace transform of the function

$$F(s) = \frac{2s+4}{(s+1)(s^2+9)}.$$

(b) Use the Laplace transform to solve the ODE

$$y'' + 4y = t,\ y(0) = 0,\ y'(0) = 1.$$

(c) Let $F(s)$ be the Laplace transform of $f(t)$. Use the convolution theorem to express the inverse Laplace transform of

$$G(s) = \frac{1}{\sqrt{4+s^2}}F(s)$$

as an integral.

(d) Explain why the function

$$F(s) = \frac{s^3+4s^2+1}{2s^6+4s^4+2s+6},$$

is a Laplace transform. (DO NOT TRY TO INVERT THIS LAPLACE TRANSFORM).

Question 4 (10 + 10 = 20 marks).

(a) Let $f(x+2) = f(x)$ for all $x \in \mathbb{R}$ and

$$f(x) = \begin{cases} 2x & 0 \leq x < 1 \\ 1 & -1 \leq x < 0. \end{cases}$$

(i) Calculate the Fourier series for f.

(ii) Denote the Fourier cosine coefficients by a_n and the sine coefficients by b_n. Determine the value of the infinite series

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(iii) To what values will the Fourier series converge at $x = 0$ and $x = 2$? Explain your answer.

(b) Consider the following boundary value problem for Laplace's equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ u(0,y) &= u(1,y) = 0, \\ u(x,0) &= x - x^2,\ u(x,1) = 0. \end{aligned}$$

(i) Obtain a solution of this problem using separation of variables and a Fourier series.

(ii) Where does the maximum value of the solution occur?

Question 5 (8 + 12 = 20 marks).

(a) (i) Show that the problem $y' = (1+x)(1+y), y(0) = 1$ has solution $y(x) = 2e^{\frac{1}{2}x(x+2)} - 1$.

(ii) Construct a second order Taylor series scheme for the initial value problem in [(i)], on the interval $[0, 1]$. Let $h = 0.1$. Now use your Taylor scheme to find approximations for the value of the solution at $x = 0.1$ and $x = 0.2$. Compare your approximate solution with the exact solution.

(b) Set up a finite difference scheme for solving the boundary value problem

$$y'' + y = x^3, \; x \in [0, 1],$$

subject to the boundary conditions $y(0) = 0, y(1) = 1$. Show that this leads to a linear system of the form $Ay = b$ where

$$A = \begin{pmatrix} h^2-2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & h^2-2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & h^2-2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & h^2-2 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}, b = \begin{pmatrix} h^2 x_1^3 \\ \vdots \\ h^2 x_{n-1}^3 - y_n \end{pmatrix}$$

Solve the resulting system in the case when $n = 4$. i.e $h = 0.25$.

END OF EXAM

Table of integrals

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$$

$$\int \frac{du}{u} = \log|u| + C$$

$$\int e^u\,du = e^u + C$$

$$\int \cos u\,du = \sin u + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \operatorname{cosech}^2 u\,du = -\coth u + C$$

$$\int \tan^2 u\,du = u - \tan u + C$$

$$\int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int u^n \ln u du = \frac{u^{1+n}}{1+n}\ln u - \frac{u^{1+n}}{(1+n)^2} + C$$

$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$$

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\frac{u}{a} + C = \log\left(u + \sqrt{a^2+u^2}\right) + C$$

$$\int \frac{du}{1-u^2} = \tanh^{-1} u + C_1 = \frac{1}{2}\log\left|\frac{1+u}{1-u}\right| + C_2$$

$$\int \cosh u\,du = \sinh u + C$$

$$\int \sinh u\,du = \cosh u + C$$

$$\int \tanh u\,du = \log\cosh u + C$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln u du = u\ln u - u + C$$

$$\int u\sin(au)du = \frac{\sin(au)}{a^2} - \frac{u\cos(au)}{a} + C$$

$$\int u\cos(au)du = \frac{\cos(au)}{a^2} + \frac{u\sin(au)}{a} + C$$

$$\int u^2\sin(au)du = \frac{2u\cos(au)}{a^2} + \frac{(a^2u^2-2)\sin(au)}{a^3}$$

$$\int u^2\cos(au)du = \frac{2u\sin(au)}{a^2} - \frac{(a^2u^2-2)\cos(au)}{a^3}.$$

Table of Laplace transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$
$$\mathcal{L}(e^{-at}) = \frac{1}{s+a}$$
$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$$
$$\mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2}$$
$$\mathcal{L}(J_0(t)) = \frac{1}{\sqrt{1+s^2}}.$$

Variation of parameters

Given that y_1 and y_2 are solutions of the ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0,$$

we seek a particular solution of the ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x),$$

by looking for solutions of the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$. The functions u and v must satisfy

$$u'y_1 + v'y_2 = 0,$$
$$u'y_1' + v'y_2' = f(x).$$

Bessel Functions

Bessel's differential equation $t^2u'' + tu' + (t^2 - \alpha^2)u = 0$ may be transformed into the equation

$$x^2y'' + (1-2s)xy' + ((s^2 - r^2\alpha^2) + a^2r^2x^{2r})y = 0$$

under the change of variables $t = ax^r$ and $y(x) = x^s u(t)$.