Q1 2010 Exam Solutions

(1) (a) $xy'' + (1-2x)y' + (x-1)y = 0$

$y_1 = e^x$ $\quad p(x) = \dfrac{1-2x}{x}$ $\quad$ (5)

$$e^{-\int p(x)\,dx} = e^{-\int(\frac{1}{x}-2)} = e^{-\ln x + 2x}$$

$$= \pm \frac{e^{2x}}{x} \cancel{\text{from}}$$

$$y_2 = y_1 \int \frac{1}{(y_1)^2} \frac{e^{2x}}{x}\,dx = e^x \ln x$$

(b) $x^2y'' - 3xy' + 3y = 0 \qquad y = x^r$

$$x^2 r(r-1)x^{r-2} - 3xrx^{r-1} + 3x^r = 0 \qquad 1\tfrac{1}{2}$$

$$y_1 = x, \quad y_2 = x^3$$

$$x^2y'' - 3xy' + 3y = 2x^4e^x$$

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$$

$\cancel{y}\ y_p = uy_1 + vy_2 \qquad W(y_1, y_2) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} \quad 1\tfrac{1}{2}$

$$= 2x^3$$

$$u' = -\frac{y_2 R}{W} = -\frac{x^3\,2x^2e^x}{2x^3} = -x^2e^x \qquad 3$$

$$u = -e^x(2 - 2x + x^2)$$

$$v' = \frac{y_1 R}{W} = \frac{x(2x^2e^x)}{2x^3} = e^x \qquad v = e^x \qquad 3$$

$$y = c_1x + c_2x^3 + 2e^x(x^2 - x)$$

(c) $1-2s=5$ $\qquad s^2-r^2\alpha^2=4$, $r^2a^2x^{2r}=-x^6$

$r=3$ $\qquad a=i/3$

$s=-2 \qquad 4-9\alpha^2=4 \qquad \alpha=0$

$$y=c_1x^{-2}I_0\left(\frac{x^3}{3}\right)+c_2x^{-2}K_0\left(\frac{x^3}{3}\right)$$

or $\quad c_1x^{-2}J_0\left(\frac{ix^3}{3}\right)+c_2x^{-2}Y_0\left(\frac{ix^3}{3}\right)$

Q2 $\quad (x^2-1)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+2x\sum_{n\geq1}^{\infty}2na_nx^n-\sum_{n=0}^{\infty}2a_nx^n$

$-2a_2-2a_0+(-6a_3+2a_1-2a_1)x$

$+\sum_{n=2}^{\infty}n(n-1)a_nx^n-\sum_{n=4}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=2}^{\infty}2na_nx^n-\sum_{n=2}^{\infty}2a_nx^n$

$a_2=a_0, \quad a_3=0$

$$\sum_{n=2}^{\infty}\left(\left(n(n-1)+(2n-2)\right)a_n-(n+2)(n+1)a_{n+2}\right)x^n=0$$

$$a_{n+2}=\cancel{n+2)}\ \frac{n^2+n-2}{(n+2)(n+1)}a_n$$

$$=\frac{n-1}{n+1}a_n$$

$a_5=\frac{3-1}{4}a_3=0 \qquad a_7=0 \quad$ etc $\quad$ odds all zero except a_1

$a_2=a_0, \quad a_4=\frac{-1}{3}a_2, \quad a_6=-\frac{3}{5}\cdot\frac{1}{3}a_2=\frac{-1}{5}a_0$

$a_8=-\frac{5}{7}\cdot\frac{1}{5}a_0=\frac{-1}{7}a_0$

$$a_{10} = -\frac{7}{9} a_8 = -\frac{1}{9} a_0$$

So $\quad a_{n+2} = -\frac{1}{n+1} a_0$

or $\quad a_{2n} = \frac{-1}{(2n-1)} a_0$

Hence $\quad y = a_1 x + a_0 \cancel{+} a_0 \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n-1)}$

(b) $\quad 2x^2 + 3xy' - (1+x)y = 0$

$$\sum_{n=0}^{\infty} 2(n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} 3(n+s) a_n x^{n+s}$$
$$- \sum_{n=0}^{\infty} a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s+1} = 0$$

$\big(2s(s-1) + 3s - 1\big) a_0 x^s, \quad 2s^2 + s - 1, \quad s = -1, \frac{1}{2}$

$$\sum_{n=1}^{\infty} \left[2(n+s)(n+s-1) + 3(n+s) - 1\right] a_n x^{n+s} - \sum_{n=0}^{\infty} a_n x^{n+s+1} = 0$$

Change starting value of first series

$$\sum_{n=0}^{\infty} \Big(\left[2(n+s+1)(n+s) + 3(n+s+1) - 1\right] a_{n+1} - a_n \Big) x^{n+s+1} = 0.$$

$$a_{n+1} = \frac{a_n}{2(n+s+1)(n+s)+3(n+s+1)-1}$$

$$= \frac{a_n}{(n+s+2)(2n+2s+1)}.$$

if

$s=-1$, $\quad a_{n+1} = \dfrac{a_n}{(n+1)(2n-1)}$.

$$a_1 = \frac{a_0}{(-1)}$$

$$a_2 = \frac{a_1}{2\times 1} = -\frac{a_1}{2 \cdot 1}$$

$$a_3 = \frac{-a_{0}}{3\times 2 \quad 1\times 3}$$

$$a_4 = \frac{-a_0}{4\times 3\times 2 \quad 1\times 3\times 5}$$

$$a_5 = \frac{-a_0}{5! \quad 1\times 3\times 5\times 7} = -\frac{2\times 4\times 6}{5! \quad 7!} \cdot a_0$$

$$= -\frac{2^3\, 3!}{5! \, 7!} a_0$$

$$a_6 = \frac{-a_0}{6! \; 1\times 3\times 5\times 7\times 9} = -\frac{2^4 4!}{6! \; 9!} a_0$$

$n\geq 2 \quad a_n = \dfrac{-a_0\, 2^{n-2}(n-2)!}{n!\,(2n-3)!}$

$$y = x^{-1}\left(a_0 - a_0 x - a_0 \sum_{n=2}^{\infty} \frac{2^{n-2}(n-2)!}{n!\,(2n-3)!} x^n\right)$$

$s=\frac{1}{2} \quad a_{n+1} = \dfrac{a_n}{(n+1)(2n+5)}$.

$$a_1 = \frac{a_0}{1\times 5}, \quad a_2 = \frac{a_0}{(2\times 1)(5\times 7)}, \quad a_3 = \frac{a_0}{3!\,(5\times 7\times 9)}$$

$$a_4 = \frac{a_0}{4!\; 5\times 7\times 9\times 11} = \frac{2\cdot 4\cdot 6\cdot 8\cdot 10\; a_0}{2\; 4!\;\; 11!}$$
$$= \frac{2\cdot 2^5\, 5!\; a_0}{2(4!\; 11!)}$$

$$a_n = \frac{2^{n+1}(n+1)!}{2\; n!\,(2n+3)!}\, a_0 = \frac{2^n (n+1)!\; a_0}{n!\,(2n+3)!}$$

$$y = a_0 x^{1/2} \sum_{n=0}^{\infty} \frac{2^n (n+1)!}{n!\,(2n+3)!} x^n$$

Q3

(a) $$F(s) = \frac{2s+4}{(s+1)(s^2+9)} = \frac{1}{5(s+1)} + \frac{11}{5(s^2+9)} - \frac{s}{5(s^2+9)}$$

$$f(t) = \frac{1}{5}e^{-t} - \frac{1}{5}\cos 3t + \frac{11}{15}\sin 3t$$

(b) $y'' + 4y = t, \quad y(0) = 0, \quad y'(0) = 1$

$$y(t) = \frac{1}{4}t + \frac{3}{8}\sin 2t$$

(c) $$\int_0^t J_0(2u)\, f(t-u)\, du$$

(d) $\frac{\deg 3}{\deg 6}$ $\quad 3 < 6$

Q4

$$f(x) = \begin{cases} 2x & 0 \le x < 1 \\ 1 & -1 \le x < 0 \end{cases}$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} \cos(n\pi x)\,dx + 2\int_0^1 x\cos(n\pi x)\,dx$$
$$= \frac{2\left(-1 + (-1)^n\right)}{n^2\pi^2}$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\int_{-1}^{0} dx + \frac{1}{2}\int_0^1 2x\,dx = \frac{1}{2} + \frac{1}{2} = 1$$

$$b_n = -\frac{\left(1 + (-1)^n\right)}{n\pi}$$

⑥

(ii) $2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \int_{-1}^{1} |f(x)|^2 dx$

$= \int_{-1}^{0} |f(x)|^2 dx + \int_0^1 |f(x)|^2 dx$

$= \int_{-1}^{0} dx + \int_0^1 4x^2 dx$

$= \frac{4}{3} + 1 = \frac{7}{3}.$

(iii) ~~$x=0,\ x=$~~ $x = 0$ $\frac{1}{2}(1 + 2\times 0) = 1/2$

by periodicity $= 1/2$ at $x=2$

(b) $u_{xx} + u_{yy} = 0$ $u(x,y) = X(x)\,Y(y)$

$u(0,y) = u(1,y) = 0 \Rightarrow X(0) = X(1) = 0$

$u(x,0) = x - x^2,$ $u(x,1) = 0 \Rightarrow Y(1) = 0$

$X_n(x) = A_n \cos(n\pi x).$ $\lambda = -n^2\pi^2$

$\frac{X''}{X} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$ $\frac{X''}{X} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda.$

$Y(y) = A\cosh(n\pi y) + B\sinh(n\pi y).$

$Y(1) = A\cosh(n\pi) + B\sinh(n\pi) = 0$

$A = -\frac{B\sinh(n\pi)}{\cosh(n\pi)}.$

$Y(y) = B\sinh(n\pi y) - \frac{B\sinh(n\pi)\cosh(n\pi y)}{\cosh(n\pi)}$

$= -\frac{B\left[\cosh(n\pi y)\sinh(n\pi) - \sinh(n\pi y)\cosh(n\pi)\right]}{\cosh(n\pi)}$

$= -B\frac{\overset{\sinh}{\cancel{\cosh}}(n\pi(y-1))}{\cosh(n\pi)}$

$$X(x)Y(y) = A_n \frac{\sinh(n\pi(1-y))}{\cosh(n\pi)} \sin(n\pi x).$$

$$\sum_{n=1}^{\infty} C_n \frac{\sinh(n\pi(1-y))}{\cosh(n\pi)} \sin(n\pi x)$$

$$C_n \tanh(n\pi) = 2\int_0^1 (x-x^2)\sin(n\pi x)\,dx$$

$$= \frac{4(1-(-1)^n)}{n^3\pi^3}.$$

$$\text{So } u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^3\pi^3 \tanh(n\pi)} \frac{\sinh(n\pi(1-y))}{\cosh(n\pi)} \sin(n\pi x)$$

Max occurs at $x = 1/2$ by Maximum principle.

(a) (i) easy, (ii) $f(x,y) = (1+x)(1+y)$

$$\frac{df}{dx} = (2+2x+x^2)(1+y).$$

$$y_{i+1} = y_i + h(1+x_i)(1+y_i) + \frac{h^2}{2}(2+2x+x^2)(1+y_i)$$

$$y_0 = 1, \quad y_1 = 1 + (0.1)(1+0)(1+1) + \frac{(0.1)^2}{2} 2(1+1)$$

$$= 1.22 \qquad \text{exact } 1.22142$$

$$y_2 = 1.22 + 0.1(1+0.1)(1+1.22)$$
$$+ \frac{(0.1)^2}{2}(2+2(0.1)+(0.1)^2)(1+1.22)$$

$$= \cancel{3.00377}\ 1.48873, \qquad \text{exact } 1.49215$$

(b) $$\begin{pmatrix} h^2-2 & 1 & 0 \\ 1 & h^2-2 & 1 \\ 0 & 1 & h^2-2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} h^2 x_1^3 \\ h^2 x_2^3 \\ h^2 x_3^3 - y_4 \end{pmatrix} \qquad y_4 = 1$$

$h = 0.25$

$$\begin{pmatrix} -1.9375 & 1 & 0 \\ 1 & -1.9375 & 1 \\ 0 & 1 & -1.9375 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.000976563 \\ 0.0078125 \\ -0.973633 \end{pmatrix}$$

$y_0 = 0, \ y_1 = 0.281212, \ y_2 = 0.545825, \ y_3 = 0.784236$