Question 1 (5 + 10 + 5 = 20 marks).

(a) The ordinary differential equation
$$x(x+1)y'' - xy' + y = 0,$$
has a solution $y_1(x) = x$. Use this to find a second linearly independent solution y_2.

(b) Use Variation of Parameters to find the general solution of the ODE
$$y'' + 4y = \tan(2x).$$
You may need $\int \sec(2x)dx = \frac{1}{2}\ln(\sec(2x) + \tan(2x))$.

(c) Solve the equation
$$x^2y'' + 4xy' + (2 + 4x^8)y = 0$$
in terms of Bessel functions.

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Question 2 (10 + 10 = 20 marks).

(a) For the ODE

$$y'' + xy' - 4y = 0.$$

obtain a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Show that of the even coefficients a_{2k}, only a_0, a_2 and a_4 are nonzero. Write the solution corresponding to these coefficients. Now obtain the first three nonzero terms of the solution involving the odd coefficients a_{2k+1}. You do not need to obtain an explicit formula for a_{2k+1} in terms of k.

(b) Consider the ODE $3xy'' + y' - y = 0$.

(i) Look for a solution of the form $y = x^s \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Show that we must have $s = 0$ or $s = 2/3$.

(ii) Show that the coefficients satisfy

$$a_{n+1} = \frac{a_n}{(n+s+1)(3n+3s+1)}, \quad n = 0, 1, 2, 3, \ldots$$

(iii) Use the recurrence relation for a_n to generate the first three nonzero terms of each of the two linearly independent solutions for the ODE. That is, the first three terms for the solutions corresponding to either $s = 0$ and $s = 2/3$. For 2 marks, write the solution corresponding to $s = 0$ or $s = 2/3$ in the form $y(x) = x^s \sum_{n=0}^{\infty} a_n(s) x^n$, with an explicit expression for the coefficient $a_n(s)$ in terms of n. (You do not have to do both).

Question 3 (5 + 9 + 4 +2 = 20 marks).

(a) Calculate the inverse Laplace transform of the function

$$F(s) = \frac{s}{(s^2+1)(s^2+4)}.$$

(b) Use the Laplace transform to solve the ODE

$$y'' - y = \cos t, \ y(0) = 0, \ y'(0) = 1.$$

The expressions

$$\frac{s}{(1+s^2)(s^2-1)} = -\frac{s}{2\,(s^2+1)} + \frac{1}{4(s-1)} + \frac{1}{4(s+1)}$$
$$\frac{1}{s^2-1} = \frac{1}{2(s-1)} - \frac{1}{2(s+1)}.$$

may be helpful.

(c) Let $F(s)$ be the Laplace transform of $f(t)$. Obtain an expression for the Laplace transform of $e^{at}f(t)$ in terms of F.

(d) Explain why the function

$$F(s) = \frac{s^3+4s^2+1}{2s^7+5s^3+2s+6},$$

is a Laplace transform. (DO NOT TRY TO INVERT THIS LAPLACE TRANSFORM).

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Question 4 (10 + 10 = 25 marks).

(a) (i) Calculate the Fourier sine series for $f(x) = x$ on the interval $[0, \pi)$.

(ii) Denote the Fourier sine coefficients by b_n. Use Parseval's formula to determine the value of the infinite series

$$\sum_{n=1}^{\infty} b_n^2.$$

(iii) To what values will the Fourier series converge at $x = 0$ and $x = \pi$? Explain your answer.

(b) Consider the following initial and boundary value problem for the heat equation.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1, \quad t > 0, \\ u(0, t) &= u(1, t) = 0, \\ u(x, 0) &= x - x^2. \end{aligned}$$

(i) By looking for solutions of the heat equation of the form $u(x, t) = X(x)T(t)$ show that the functions X and T must satisfy the problems

$$X''(x) = \lambda X(x), \quad X(0) = X(1) = 0,$$

and

$$T'(t) = \lambda T(t)$$

for some constant λ. Show that the only values of λ which lead to nonzero solutions of the differential equation for X, satisfying the given conditions, are $\lambda = -n^2\pi^2$, for $n = 1, 2, 3, \ldots$. Hence obtain expressions for X and T.

(ii) Express the solution of the given problem for the heat equation as an infinite series involving the Fourier sine or cosine coefficients of $u(x, 0)$. You must explicitly calculate the relevant Fourier coefficients.

Question 5 (10 + 10 = 20 marks).

(a) Construct a second order Taylor series scheme for solving the nonlinear initial value problem

$$y' = xy^2, y(0) = 1,$$

on the interval $[0, 1]$. Let $h = 0.1$. Now use your Taylor scheme to find approximations for the value of the solution at $x = 0.1$ and $x = 0.2$. Verify that the exact solution is $y(x) = \dfrac{2}{2 - x^2}$. Compare the exact and approximate solutions at $x = 0.1, 0.2$ and find the error of the approximation at each point.

(b) Set up a finite difference scheme for solving the boundary value problem

$$y'' - 4y = x, \ x \in [0, 1],$$

subject to the boundary conditions $y(0) = 0, y(1) = 1$. Show that this leads to a linear system of the form $Ay = b$ where A is the tridiagonal matrix

$$A = \begin{pmatrix} -4h^2 - 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -4h^2 - 2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -4h^2 - 2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & -4h^2 - 2 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}, \quad b = \begin{pmatrix} h^2 x_1 \\ \vdots \\ h^2 x_{n-1} - 1 \end{pmatrix}.$$

Solve the resulting system in the case when $n = 4$. i.e $h = 0.25$. You may need the approximation

$$y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2},$$

where $y_i = y(x_i)$.

END OF EXAM

Table of integrals

$$\int u^n\,du = \frac{u^{n+1}}{n+1}$$

$$\int \frac{du}{u} = \log|u|$$

$$\int e^u\,du = e^u$$

$$\int \cos u\,du = \sin u$$

$$\int \sin u\,du = -\cos u$$

$$\int \operatorname{cosech}^2 u\,du = -\coth u$$

$$\int \tan^2 u\,du = u - \tan u$$

$$\int \sec u \tan u\,du = \sec u$$

$$\int \csc u \cot u\,du = -\csc u$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$$

$$\int u^n \ln u du = \frac{u^{1+n}}{1+n}\ln u - \frac{u^{1+n}}{(1+n)^2}$$

$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u$$

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\frac{u}{a} = \log\left(u+\sqrt{a^2+u^2}\right)$$

$$\int \frac{du}{1-u^2} = \tanh^{-1} u = \frac{1}{2}\log\left|\frac{1+u}{1-u}\right|$$

$$\int \cosh u\,du = \sinh u$$

$$\int \sinh u\,du = \cosh u$$

$$\int \tanh u\,du = \log\cosh u$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln u du = u\ln u - u$$

$$\begin{aligned}
\int u\sin(au)du &= \frac{\sin(au)}{a^2} - \frac{u\cos(au)}{a}\\
\int u\cos(au)du &= \frac{\cos(au)}{a^2} + \frac{u\sin(au)}{a}\\
\int u^2\sin(au)du &= \frac{2u\sin(au)}{a^2} - \frac{(a^2u^2-2)\cos(au)}{a^3}\\
\int u^2\cos(au)du &= \frac{2u\cos(au)}{a^2} + \frac{(a^2u^2-2)\sin(au)}{a^3}.
\end{aligned}$$

Table of Laplace transforms

$$\begin{aligned}
\mathcal{L}(t^n) &= \frac{n!}{s^{n+1}} \\
\mathcal{L}(e^{-at}) &= \frac{1}{s+a} \\
\mathcal{L}(\sin(at)) &= \frac{a}{s^2+a^2} \\
\mathcal{L}(\cos(at)) &= \frac{s}{s^2+a^2} \\
\mathcal{L}(J_0(t)) &= \frac{1}{\sqrt{1+s^2}}.
\end{aligned}$$

Variation of parameters

Given that y_1 and y_2 are solutions of the ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0,$$

we seek a particular solution of the ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x),$$

by looking for solutions of the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$. The functions u and v must satisfy

$$\begin{aligned}
u'y_1 + v'y_2 &= 0, \\
u'y_1' + v'y_2' &= f(x).
\end{aligned}$$

Bessel Functions

Bessel's differential equation $t^2u'' + tu' + (t^2 - \alpha^2)u = 0$ may be transformed into the equation

$$x^2y'' + (1-2s)xy' + ((s^2 - r^2\alpha^2) + a^2r^2x^{2r})y = 0$$

under the change of variables $t = ax^r$ and $y(x) = x^s u(t)$.

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)dx, \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \\
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.
\end{aligned}$$