Exam 2014, DES Solutions

1) (a) $x(x+1)y'' - xy' + y = 0 \qquad y'' - \frac{1}{x+1}y' + \frac{1}{x(x+1)}y = 0$

$p(x) = -\frac{1}{x+1}, \qquad y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx}$

$-\int p(x)\,dx = \ln(x+1)$

So $e^{-\int p(x)} = e^{\ln(x+1)} = x+1$ and $y_2 = x\int \frac{1}{x^2}(x+1)dx$

$= x\int\left(\frac{1}{x} + \frac{1}{x^2}\right)dx = x\left(\ln x - \frac{1}{x}\right)$

$= x\ln x - 1$

(b) $y'' + 4y = \tan(2x). \qquad y_1 = \sin(2x),\ y_2 = \cos(2x).$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin(2x) & \cos(2x) \\ 2\cos(2x) & -2\sin(2x) \end{vmatrix} = 2(-\sin^2(2x) - \cos^2(2x))$
$= -2.$

$y_p = uy_1 + vy_2, \qquad u' = -\frac{y_2 R}{W}, \quad v' = \frac{y_1 R}{W}$

$u = -\int \frac{\cos(2x)\tan(2x)}{-2}dx = \frac{1}{2}\int \sin(2x)dx$

$= -\frac{1}{4}\cos(2x)$

$v' = \frac{-1}{2}\int \frac{\sin(2x)\sin(2x)}{\cos(2x)}dx = -\frac{1}{2}\int \frac{\sin^2(2x)}{\cos(2x)}dx$

$= -\frac{1}{2}\int \frac{1-\cos^2(2x)}{\cos(2x)}dx = -\frac{1}{2}\int(\sec(2x) - \cos(2x))dx$

$= -\frac{1}{4}\ln(\sec(2x) + \tan(2x)) + \frac{1}{4}\sin(2x)$

So $y_p = -\frac{1}{4}\cos(2x)\sin(2x) + \cos(2x)\left(\frac{1}{4}\sin(2x) - \frac{1}{4}\ln(\sec(2x)+\tan(2x))\right)$

$y = C_1\cos(2x) + C_2\sin(2x) - \frac{\cos(2x)}{4}\ln(\sec(2x)+\tan(2x))$

(5) $x^2y'' + 4xy' + (2+4x^8)y = 0$

$1-2s = 4, \quad s^2 - r^2\alpha^2 = 2, \quad a^2r^2x^{2r} = 4x^8$

$s = -\frac{3}{2}, \quad r = 4, \quad \cancel{4\alpha^2} = \cancel{4}, \quad 16a^2 = 4, \quad a = \frac{1}{2}$

$\frac{9}{4} - 16\alpha^2 = 2, \quad \frac{9}{4} - 2 = 16\alpha^2$

$16\alpha^2 = \frac{1}{4} \quad \therefore \alpha = \pm\frac{1}{8}$

So $y = c_1 x^{-3/2} J_{\frac{1}{8}}\left(\frac{x^4}{2}\right) + c_2 x^{-3/2} J_{-\frac{1}{8}}\left(\frac{x^4}{2}\right)$.

2) $y'' + xy' - 4y = 0. \quad y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 4 a_n x^n$

$= \cancel{\sum_{n=2}^{\infty} 2a_2 - 4a_0}$

$2a_2 - 4a_0 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} 4 a_n x^n$

$= 2a_2 - 4a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n-4) a_n\right] x^n$

$a_2 = 2a_0, \quad a_{n+2} = \frac{-(n-4)}{(n+2)(n+1)} a_n$

Now $a_4 = \frac{-(-2)}{4\times 3} a_2 = +\frac{1}{3} a_0$

$a_6 = \frac{-(4-4)}{6\times 5} a_4 = 0$ all other even coefficients are non zero.

So

$y_1 = a_0\left(1 + 2x^2 + \frac{1}{3}x^4\right)$

Odd coefficients

$$a_3 = \frac{-(1-4)a_1}{3\times 2} = \frac{1}{2}a_1,$$

$$a_5 = \frac{-(3-4)a_3}{5\times 4} = \frac{1}{5\times 4\times 2}a_1 = \frac{1}{40}a_1$$

So $y = a_1\left(x + \frac{1}{2}x^3 + \frac{1}{40}x^5 + \cdots\right)$

b) $3xy'' + y' - y = 0$, $y = \sum_{n=0}^{\infty} a_n x^{n+s}$

$$\sum_{n=0}^{\infty} 3(n+s)(n+s-1)a_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$= \big(3s(s-1)+s\big)a_0 x^{s-1} + \sum_{n=1}^{\infty}\big[3(n+s)(n+s-1)+(n+s)\big]a_n x^{n+s-1}$$

$$- \sum_{n=0}^{\infty} a_n x^{n+s}$$

$3s^2 - 2s = 0$, $s = 0, 2/3$

and $\sum_{n=1}^{\infty}\Big\{\big[3(n+s)(n+s-1)+(n+s)\big]a_n - a_{n-1}\Big\}x^{n+s-1} = 0$

$$a_n = \frac{a_{n-1}}{3(n+s)(n+s-1)+(n+s)} \qquad n\geq 1$$

or $$a_{n+1} = \frac{a_n}{(n+s+1)(3n+3s+1)} \qquad n\geq 0$$

$s=0$, $a_{n+1} = \dfrac{a_n}{(n+1)(3n+1)}$. $a_1 = a_0$, $a_2 = \dfrac{a_1}{2\times 4}$, $a_3 = \dfrac{a_2}{3\times 7}$

$= \dfrac{a_0}{1\times 2\times 3\times 1\times 4\times 7}$ $a_n = \dfrac{a_0}{n!(1\times 4\times\cdots(3n-2))}$ $n\geq 1$

$s=\frac{2}{3}$, $a_{n+1} = \dfrac{a_n}{(n+1)(3n+5)}$, $a_1 = \dfrac{a_0}{5}$, $a_2 = \dfrac{a_0}{2!\,5\times 8}$, ...

$a_n = \dfrac{a_0}{n!\,5\times 8\times\cdots(3n+2)}$

$$y = A\left(1+\sum_{n=1}^{\infty}\frac{x^n}{n!(1\times 4\times\ldots(3n-2))}\right) + Bx^{2/3}\sum_{n=0}^{\infty}\frac{x^n}{n!\,5\times\ldots(3n+2)}$$

3 (a) $$F(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{s}{1+s^2} - \frac{s}{s^2+4}\right]$$

$$\mathcal{L}^{-1}\left(F(s)\right) = \frac{1}{3}\left(\cos t - \cos(2t)\right)$$

(b) $\cancel{(s^2-1)Y(s) = \frac{s}{1+s^2}}$ $\quad y'' - y = \cos t,\ y(0)=0,\ y'(0)=1$

$$\left(s^2 Y(s) - s y(0) - y'(0)\right) - Y(s) = \frac{s}{1+s^2},\ \cancel{y}$$

$$(s^2-1)Y(s) = \frac{s}{1+s^2} + 1$$

$$Y(s) = \frac{s}{(1+s^2)(s^2-1)} + \frac{1}{s^2-1}$$

$$= \frac{-s}{2(1+s^2)} + \frac{1}{4}\left(\frac{1}{s-1} + \frac{1}{s+1}\right) + \frac{1}{2}\left(\frac{-1}{s+1} + \frac{1}{s-1}\right)$$

$\cancel{y(t) = -\frac{1}{2}\cos t + \frac{3}{4}(e^{-t} + e^{t})}$

$$y(t) = -\frac{1}{2}\cos t + \frac{3}{4}e^{t} - \frac{1}{4}e^{-t}$$

c) $$\int_0^{\infty} e^{at} f(t) e^{-st}\,dt = \int_0^{\infty} f(t) e^{-(s-a)t}\,dt = F(s-a)$$

(d) Ratio of two polynomials, degree num < deg denom

Q4

a)(i) $\frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \frac{2}{\pi}\left(\left[\frac{-x\cos(nx)}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)\,dx}{n}\right)$

$= \frac{2(-1)^{n+1}}{n}$

(ii) $\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{n=1}^{\infty}\frac{4}{n^2}$

$\frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$

$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$

(iii) Converges to 0 at 0, $\frac{1}{2}\left[f(\pi^+)+f(\pi^-)\right] = \frac{1}{2}\left[\pi-\pi\right] = 0$

(b) Book work

$\cancel{\frac{2}{\pi}\int_0^{\pi}(x-x^2)\sin(nx)\,dx} = b_n = 2\int_0^1 (x-x^2)\sin(n\pi x)\,dx$

$= \frac{4(1-(-1)^n)}{n^3\pi^3}$

$u(x,t) = 4\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^3\pi^3}\sin(n\pi x)\,e^{-n^2\pi^2 t}.$

Q5 $y' = xy^2.$

$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = y^2(x) + 2y(x)\,x\,y'(x)$

$= y^2(x) + 2y(x)\,x\,(xy^2)$

$= y^2(x) + 2x^2y^3(x)$

$y_{i+1} = y_i + hx_iy_i^2 + \frac{h^2}{2}\left[y_i^2 + 2x_i^2y_i^3\right],\ y_0 = 1$

$y_1 = 1.005,\ y_2 = 1.02025$

true $= 1.00503$ true 1.02041

$$y'' - 4y = x \qquad y(0) = 0, \; y(1) = 1.$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - 4y_i = x_i \qquad i = 1, \ldots, n-1$$

$$y_{i+1} - 2y_i + y_{i-1} - 4h^2 y_i = h^2 x_i$$

$$\begin{pmatrix} -4h^2-2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -4h^2-2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -4h^2-2 & 1 & & \\ & & & & & \\ \vdots & & & & \cdots & -4h^2-2 \end{pmatrix} \begin{pmatrix} y_1 \\ \\ \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} h^2 x_i \\ \\ \\ h^2 x_{n-1} - 1 \end{pmatrix}$$

$n = 4$

$$\begin{pmatrix} -2.25 & 1 & 0 \\ 1 & -2.25 & 1 \\ 0 & 1 & -2.25 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.015625 \\ 0.03125 \\ -.953125 \end{pmatrix}$$

$$y_1 = 0.118906$$
$$y_2 = 0.283163$$
$$y_3 = 0.549461$$