Question 1 (5 + 10 + 5 = 20 marks).

(a) The ordinary differential equation

$$xy'' + (1 + 2x)y' + (x + 1)y = 0,$$

has a solution $y_1(x) = e^{-x}$. Use this to find a second linearly independent solution y_2.

(b) Use Variation of Parameters to solve

$$x^2y'' + 3xy' - 3y = x^2 \ln x.$$

Recall solutions of an Euler type equation are of the form $y = x^a$.

(c) Solve

$$x^4y'' + 3x^3y' + (5x^2 - 1)y = 0$$

in terms of Bessel functions. Note that when putting the equation into the form of a general Bessel equation, the constant r does not have to be positive.

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Question 2 (10 + 10 = 20 marks).

(a) Consider the ODE

$$y'' - x^2 y' - 3xy = 0.$$

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Show that we must have $a_2 = 0$ and

$$a_{3k} = \frac{a_0}{2 \cdot 5 \cdot 8 \cdots (3k-1)}, \quad k \geq 1, \quad a_{3k+1} = \frac{a_1}{3^k k!}, \quad k \geq 0.$$

Obtain the general solution and identify the solution arising from the a_{3k+1} terms as an explicit function. (Hint: Write the solution for the a_{3k+1} terms in the form $y = x \sum_{k=0}^{\infty} a_{3k+1} (x^3)^k$. See information at end of exam).

(b) Consider the ODE

$$2xy'' + y' - \beta y = 0.$$

Here β is a positive real number.

(i) Look for a solution of the form $y = x^s \sum_{n=0}^{\infty} a_n x^n$. Show that we must have $s(2s-1) = 0$.

(ii) Show that the coefficients satisfy

$$a_n = \frac{\beta a_{n-1}}{(n+s)(2n+2s-1)}, \quad n \geq 1.$$

(iii) Use the recurrence relation for a_n to generate two linearly independent solutions for the ODE.

Question 3 (5 + 10 + 2 + 3 = 20 marks).

(a) Calculate the inverse Laplace transform of the function

$$F(s) = \frac{s+7}{(s^2+9)(s^2+4)}.$$

(b) Use the Laplace transform to solve the ODE

$$y'' - 16y = \cos t, \;\; y(0) = 2, \;\; y'(0) = 1.$$

(c) Let $F(s)$ be the Laplace transform of $f(t)$. Obtain the Laplace transform of $f(at-1)H(at-1)$ in terms of $F(s)$, where H is the Heaviside step function.

(d) Calculate the inverse Laplace transform of $F(s) = \dfrac{1}{s} \tanh^{-1}\left(\dfrac{1}{s^2}\right)$ as a series. You may need

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots .$$

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Question 4 (7+3+ 10 = 20 marks).

(a) Let

$$f(x) = \begin{cases} x & 0 \le x \le L/2 \\ L - x & L/2 < x \le L. \end{cases}$$

(i) Calculate the Fourier cosine series for f on $[0, L)$.

(ii) To what values will the Fourier cosine series converge at $x = L/2$ and $x = -L/2$? Explain your answer.

(b) You are given that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(nx)}{n^3} = \frac{1}{12}\left(\pi^2 x - x^3\right)$$

for $-\pi < x < \pi$. Use this and Parseval's Theorem to calculate the value of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^6}.$$

(c) Consider the following initial and boundary value problem for the wave equation.

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \quad t > 0,$$
$$u(0,t) = u(1,t) = 0,$$
$$u(x,0) = x(1-x), \; u_t(x,0) = 1.$$

(i) By looking for solutions of the wave equation of the form $u(x,t) = X(x)T(t)$ show that the functions X and T must satisfy the problems

$$X''(x) = \lambda X(x), \quad X(0) = X(1) = 0,$$

and

$$T''(t) = \lambda c^2 T(t)$$

for some constant λ. Show that the only values of λ which lead to nonzero solutions of the differential equation for X, satisfying the given conditions are $\lambda = -n^2\pi^2$, for $n = 1, 2, 3, ...$ Hence obtain expressions for X and T.

(ii) Obtain the explicit solution of the PDE as the sum of two Fourier series.

Question 5 (10 + 10 = 20 marks).

(a) Construct a second order Taylor series scheme for solving the nonlinear initial value problem

$$y' = -(2 + x^2)y^2, \quad y(0) = 1,$$

on the interval $[0, 1]$. Let $h = 0.1$. Now use your Taylor scheme to find approximations for the value of the solution at $x = 0.1$ and $x = 0.2$. Solve the initial value problem exactly and determine the accuracy of your answer.

(b) Set up a finite difference scheme for solving the boundary value problem

$$y'' - 4y = \frac{1}{2}x, \; x \in [0, 1],$$

subject to the boundary conditions $y(0) = 1, y(1) = 1$. Show that this leads to a linear system of the form $Ay = b$ where

$$A = \begin{pmatrix} -4h^2 - 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -4h^2 - 2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -4h^2 - 2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & -4h^2 - 2 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}, b = \begin{pmatrix} \frac{1}{2}h^2 x_1 - y_0 \\ \vdots \\ \frac{1}{2}h^2 x_{n-1} - y_n \end{pmatrix}$$

Solve the resulting system in the case when $n = 4$. i.e $h = 0.25$. You may need the approximation

$$y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2},$$

where $y_i = y(x_i)$.

END OF EXAM

Table of integrals

$$\int u^n\,du = \frac{u^{n+1}}{n+1}$$

$$\int \frac{du}{u} = \log|u|$$

$$\int e^u\,du = e^u$$

$$\int \cos u\,du = \sin u$$

$$\int \sin u\,du = -\cos u$$

$$\int \operatorname{cosech}^2 u\,du = -\coth u$$

$$\int \tan^2 u\,du = u - \tan u$$

$$\int \sec u \tan u\,du = \sec u$$

$$\int \csc u \cot u\,du = -\csc u$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$$

$$\int u^n \ln u du = \frac{u^{1+n}}{1+n}\ln u - \frac{u^{1+n}}{(1+n)^2}$$

$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u$$

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\frac{u}{a} = \log\left(u+\sqrt{a^2+u^2}\right)$$

$$\int \frac{du}{1-u^2} = \tanh^{-1} u = \frac{1}{2}\log\left|\frac{1+u}{1-u}\right|$$

$$\int \cosh u\,du = \sinh u$$

$$\int \sinh u\,du = \cosh u$$

$$\int \tanh u\,du = \log\cosh u$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln u du = u\ln u - u$$

$$\int \sec u\,du = \ln(\sec u + \tan u)$$

$$\int u\sin(au)du = \frac{\sin(au)}{a^2} - \frac{u\cos(au)}{a}$$

$$\int u\cos(au)du = \frac{\cos(au)}{a^2} + \frac{u\sin(au)}{a}$$

$$\int u^2\sin(au)du = \frac{2u\sin(au)}{a^2} - \frac{(a^2u^2-2)\cos(au)}{a^3}$$

$$\int u^2\cos(au)du = \frac{2u\cos(au)}{a^2} + \frac{(a^2u^2-2)\sin(au)}{a^3}.$$

Table of Laplace transforms

$$
\begin{aligned}
\mathcal{L}(t^n) &= \frac{n!}{s^{n+1}} \\
\mathcal{L}(e^{-at}) &= \frac{1}{s+a} \\
\mathcal{L}(\sin(at)) &= \frac{a}{s^2+a^2} \\
\mathcal{L}(\cos(at)) &= \frac{s}{s^2+a^2} \\
\mathcal{L}(J_0(t)) &= \frac{1}{\sqrt{1+s^2}}.
\end{aligned}
$$

Variation of parameters

Given that y_1 and y_2 are solutions of the ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0,$$

we seek a particular solution of the ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x),$$

by looking for solutions of the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$. The functions u and v must satisfy

$$
\begin{aligned}
u'y_1 + v'y_2 &= 0, \\
u'y_1' + v'y_2' &= f(x).
\end{aligned}
$$

Bessel Functions

Bessel's differential equation $t^2u'' + tu' + (t^2 - \alpha^2)u = 0$ may be transformed into the equation

$$x^2y'' + (1-2s)xy' + ((s^2 - r^2\alpha^2) + a^2r^2x^{2r})y = 0$$

under the change of variables $t = ax^r$ and $y(x) = x^s u(t)$.

Parseval's Theorem gives the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

The exponential function satisfies $e^z = \displaystyle\sum_{k=0}^{\infty} z^k/k!$.