Exam Solutions DES 2015

Q1 (a) $xy'' + (1+2x)y' + (x+1)y = 0$

$y = e^{-x}$ is a solution. divide by x to get

$$y'' + \left(\frac{1+2x}{x}\right)y' + \frac{(x+1)}{x}y = 0$$

$$p(x) = \frac{1+2x}{x}.$$

A second solution is $y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{(y_1)^2}\,dx$

$$-\int p(x)\,dx = -\int\left(\frac{1}{x}+2\right)dx = -\ln x - 2x$$

So $$e^{-\int p(x)\,dx} = e^{-\ln x - 2x} = \frac{1}{x}e^{-2x}$$

and $$y_2 = e^{-x}\int \frac{1}{x}\frac{e^{-2x}}{(e^{-2x})}\,dx = \underline{\underline{e^{-x}\ln x}}.$$

(b) $x^2y'' + 3xy' - 3y = x^2 \ln x$. $y_c = x^a$

So
$$x^2 a(a-1)x^{a-2} + 3xax^{a-1} - 3x^a$$
$$= x^a(a^2 + 2a - 3)$$

$a = 1, -3$ $\therefore$ $x^2y'' + 3xy' - 3y = 0$ has solutions

$$y_1 = x,\quad y_2 = \frac{1}{x^3}$$

$y'' + \frac{3}{x}y' - \frac{3}{x^2}y = \ln x$ $\therefore$ $R(x) = \ln x$

$$W(y_1, y_2) = \begin{vmatrix} x & x^{-3} \\ 1 & -3x^{-4} \end{vmatrix} = -4x^{-3}$$

$y_p = uy_1 + vy_2$. $u = -\int \frac{y_2 R}{W} = -\int \frac{x^{-3}\ln x}{-4x^{-3}}$

$$= \frac{1}{4}\int \ln x\,dx = \frac{1}{4}(x\ln x - x)$$

$$v = \int \frac{y_1 R}{W} = -\frac{1}{4}\int \cancel{x}x^3 \ln x = -\frac{1}{4}\int x^4 \ln x\, dx$$

$$= \frac{x^5}{100} - \frac{1}{20} x^5 \ln x$$

$$\therefore\ y_p = u y_1 + v y_2 = \frac{x}{4}\left(x \ln x - x\right) + x^{-3}\left(\frac{x^5}{100} - \frac{x^5}{20}\ln x\right)$$

$$= \frac{1}{5} x^2 \ln x - \frac{6}{25} x^2$$

c) $$x^4 y'' + 3x^3 y' + (5x^2 - 1) y = 0$$

So $$x^2 y'' + 3x y' + \left(5 - \frac{1}{x^2}\right) y = 0$$

$$1 - 2s = 3, \quad s^2 - r^2 \alpha^2 = 5, \quad a^2 r^2 x^{2r} = -\frac{1}{x^2}$$

$$\therefore\ r = -1, \quad a^2 = -1, \quad \therefore\ a = i$$

$$1 - 2s = 3 \quad \therefore\ -2s = 2, \quad s = -1$$

$$1 - \alpha^2 = 5, \quad \alpha^2 = -4 \quad \alpha = \pm 2i$$

$$\therefore\ y = c_1 x^{-1} J_{2i}\left(i \frac{1}{x}\right) + c_2 J_{-2i}\left(i \frac{1}{x}\right)$$

$$= \frac{1}{x}\left(c_1 I_{2i}\left(\frac{1}{x}\right) + c_2 I_{-2i}\left(\frac{1}{x}\right)\right).$$

Q2 $y'' - x^2 y' - 3xy = 0$. Put $y = \sum_{n=0}^{\infty} a_n x^n$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} - 3x \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} 3 a_n x^{n+1}$$

$$= \cancel{2a_2 + \sum_{n=1}^{\infty} 3}\ 2a_2 + (6a_3 - 3a_0)x + \sum_{n=4}^{\infty} n(n-1) a_n x^{n-2}$$

$$- \sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=1}^{\infty} 3 a_n x^{n+1}$$

$$= 2a_2 + (6a_3 - 3a_0)x + \sum_{n=1}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1}$$

$$- \sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=1}^{\infty} 3 a_n x^{n+1}$$

$$= 2a_2 + (6a_3 - 3a_0)x + \sum_{n=1}^{\infty} \left[(n+3)(n+2) a_{n+3} - (n+3) a_n\right] x^{n+1} = 0$$

$\therefore \quad a_2 = 0 \quad a_3 = \frac{1}{2} a_0$

$$a_{n+3} = \frac{a_n}{n+2} \qquad n \geq 1.$$

Clearly $a_5 = 0$, $a_8 = 0$ etc

Take $n = 3$ $\quad a_6 = \frac{a_3}{5} = \frac{a_0}{2.5}$, $a_9 = \frac{a_0}{2.5.8}$ etc

$$a_{3k} = \frac{a_0}{2.5.8.\ (3k-1)} \qquad k \geq 1.$$

$n = 1$ $\quad a_4 = \frac{a_1}{3}$. $\quad a_7 = \frac{a_4}{6} = \frac{a_1}{3.6}$

$$a_{10} = \frac{a_7}{9} = \frac{a_1}{3.6.9} = \frac{a_1}{3^3\, 3!} \quad \text{etc}$$

$$\therefore \quad a_{3k+1} = \frac{a_1}{3^k\, k!}$$

$\therefore$

$$y = a_0\left(1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2.5.8\ldots(3n-1)}\right) + a_1 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{n!\,3^n}$$

and $$a_1 x \sum_{n=0}^{\infty} \frac{(x^3)^n/3^n}{n!} = a_1 x \sum_{n=0}^{\infty} \frac{(x/3)^n}{n!} = a_1 x e^{x^3/3}.$$

(b) $2xy'' + y' - \beta y = 0 \qquad y = x^s \sum_{n=0}^{\infty} a_n x^n$

$$2x \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} - \beta \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$= \sum_{n=0}^{\infty} 2(n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} - \sum_{n=0}^{\infty} \beta a_n x^{n+s}$$

$$= (2s(s-1) + s) a_0 x^{s-1} + \sum_{n=1}^{\infty} 2(n+s)(n+s-1) a_n x^{n+s-1} + \sum_{n=1}^{\infty} (n+s) a_n x^{n+s-1} - \sum_{n=0}^{\infty} \beta a_n x^{n+s}$$

$$= (2s-1)s a_0 x^{s-1} + \sum_{n=1}^{\infty} \left[(2(n+s)(n+s-1) + (n+s))a_n - \beta a_{n-1}\right] x^{n+s-1} = 0$$

$\therefore \ (2s-1)s = 0, \qquad a_n = \dfrac{\beta a_{n-1}}{(n+s)(2n+2s-1)}, \ n \geq 1.$

$s = 0, \ 1/2$

Take $s=0$ $\qquad a_n = \dfrac{\beta a_{n-1}}{n(2n-1)} \qquad n \geq 1$

$$a_1 = \frac{\beta a_0}{1 \times 1}, \quad a_2 = \frac{\beta a_1}{2 \times 3} = \frac{\beta^2 a_0}{1 \times 2 \times 3}$$

$$a_3 = \frac{\beta a_2}{3 \times 5} = \frac{\beta^3 a_0}{(1 \times 2 \times 3)(1 \times 3 \times 5)}$$

$$a_4 = \frac{\beta a_3}{4 \times 7} = \frac{\beta^4 a_0}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)} \qquad \text{etc}$$

$$a_n = \frac{\beta^n a_0}{n!\,(1 \times 3 \times \cdots (2n-1))} \qquad n \geq 1.$$

$$\therefore \ y = a_0\left(1 + \sum_{n=0}^{\infty} \frac{(\beta x)^n}{n!\,(1 \times 3 \times \cdots (2n-1))}\right)$$

$s = 1/2 \qquad a_n = \dfrac{\beta a_{n-1}}{2n(n+\frac{1}{2})} = \dfrac{\beta a_{n-1}}{n(2n+1)}, \quad n \geq 1$

$a_1 = \dfrac{\beta_0 a_0}{1 \times 3}, \quad a_2 = \dfrac{\beta a_1}{2(5)} = \dfrac{\beta^2 a_0}{1 \times 2 \times (3 \times 5)}$

$a_3 = \dfrac{\beta a_2}{3(7)} = \dfrac{\beta^3 a_0}{(1 \times 2 \times 3)(3 \times 5 \times 7)}$

etc. $\quad a_n = \dfrac{\beta^n a_0}{n!(1 \times 3 \times \cdots (2n+1))} \quad n \geq 0$

$\therefore \; y = x^{1/2} \displaystyle\sum_{n=0}^{\infty} \frac{(\beta x)^n}{n!(1 \times 3 \times \cdots (2n+1))}$ is second solution.

3) a) $F(s) = \dfrac{s+7}{(s^2+9)(s^2+4)} = \dfrac{As+B}{s^2+9} + \dfrac{Cs+D}{s^2+4}$

$= \dfrac{7}{5(s^2+4)} + \dfrac{s}{5(s^2+4)} - \dfrac{7}{5(s^2+9)} - \dfrac{s}{5(s^2+9)}$

$\therefore \; \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\dfrac{7}{5} \cdot \dfrac{1}{2} \dfrac{2}{s^2+4}\right) + \mathcal{L}^{-1}\left(\dfrac{1}{5} \dfrac{s}{s^2+4}\right)$

$- \mathcal{L}^{-1}\left(\dfrac{7}{5} \dfrac{1}{3} \dfrac{3}{s^2+9}\right) - \mathcal{L}^{-1}\left(\dfrac{1}{5} \dfrac{s}{s^2+9}\right)$

$= \dfrac{7}{10} \sin(2t) + \dfrac{1}{5} \cos(2t) - \dfrac{7}{15} \sin(3t) - \dfrac{1}{5} \sin(3t)$

b) $y'' - 16y = \cos t, \quad y(0) = 2, \; y'(0) = 1$. Take L.T in t

$s^2 Y(s) - s y(0) - y'(0) - 16 Y(s) = \dfrac{s}{s^2+1}$

So $(s^2 + 16) Y(s) = \dfrac{s}{s^2+1} + 2s + 1$

$\cancel{Y(s) = \dfrac{s}{(s^2+1)(s^2-16)} + \dfrac{2s+1}{s^2-16}}$

$$Y(s) = \frac{s}{(s^2+1)(s^2-16)} + \frac{2s+1}{s^2-16}$$

$$= \frac{1}{34}\left\{\frac{1}{s+4} + \frac{1}{s-4}\right\} - \frac{s}{17(s^2+1)} + \frac{9}{8(s-4)} + \frac{7}{8(s+4)}$$

$$\therefore \mathcal{L}^{-1}(Y(s)) = \left(\frac{1}{34}+\frac{9}{8}\right)e^{4t} + \left(\frac{1}{34}+\frac{7}{8}\right)e^{-4t} - \frac{1}{17}\cos t$$

$$= \frac{123}{136}e^{-4t} + \frac{157}{136}e^{4t} - \frac{1}{17}\cos t$$

c) $\int_0^\infty f(at-1)H(at-1)e^{-st}\,dt \qquad at = u$

$$= \int_0^\infty f(u-1)H(u-1)e^{-s\frac{u}{a}}\,\frac{du}{a}$$

$$= \int_1^\infty f(u-1)e^{-su/a}\,\frac{du}{a} \qquad u-1 = z$$

$$= \int_0^\infty f(z)e^{-s\frac{(z+1)}{a}}\,\frac{dz}{a} = \frac{1}{a}e^{-\frac{s}{a}}F\left(\frac{s}{a}\right)$$

d) $F(s) = \frac{1}{s}\left(\frac{1}{s^2} + \frac{1}{3}\frac{1}{s^6} + \frac{1}{5}\frac{1}{s^{10}} + \frac{1}{7}\frac{1}{s^{14}} + \cdots\right)$

$$= \frac{1}{s^3} + \frac{1}{3}\frac{1}{s^7} + \frac{1}{5}\frac{1}{s^{11}} + \frac{1}{7}\frac{1}{s^{15}} + \cdots$$

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) + \mathcal{L}^{-1}\left(\frac{1}{3s^7}\right) + \mathcal{L}^{-1}\left(\frac{1}{5s^{11}}\right) + \cdots$$

$$= \frac{1}{2}t^2 + \frac{1}{2160}t^6 + \cdots \frac{t^{4n+2}}{(2n+1)(4n+2)!} + \cdots$$

Q4 a) i) $f(x) = \begin{cases} x & 0 \le x \le L/2 \\ L-x & \frac{L}{2} < x \le L \end{cases}$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^{L/2} x\,dx + \frac{1}{L}\int_{L/2}^{L} (L-x)\,dx$$

$$= L/4$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{L}\left(\int_0^{L/2} x\cos\left(\frac{n\pi}{L}x\right)dx\right) + \frac{2}{L}\int_{L/2}^{L}(L-x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{L}{n^2\pi^2}\left(n\pi\sin\left(\frac{n\pi}{2}\right) + 2\left(\cos\left(\frac{n\pi}{2}\right)-1\right)\right)$$

$$- \frac{L}{n^2\pi^2}\left(n\pi\sin\left(\frac{n\pi}{2}\right) + 2\cos(n\pi) - 2\cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{8L}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)\sin^2\left(\frac{n\pi}{4}\right)$$ [This requires work and is not necessary]

(ii) By Dirichlet, series converges to $L/2$ at $x = L/2$. Since cosine series is even series converges to $f(x)$ at $-x$. ie. series also converges to $L/2$

(b) $$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{12}\left(\pi^2 x - x^3\right)^2 dx = \frac{\pi^6}{945}.$$

(c) This is in lecture notes

5) (a) $y' = -(2+x^2)y^2, \; y(0) = 1$

Separation of variables gives $y = \dfrac{3}{x^3 + 6x + 3}$

(a) $f(x,y) = -(2+x^2)y^2$

$$y(x+h) = y(x) + h\,y'(x) + \frac{h^2}{2} y''(x)$$

$$y' = \frac{d}{dx} f(x,y) = -2xy^2 - 2(2+x^2)y\frac{dy}{dx}$$

$$= -\cancel{2xy^2 \ldots}$$

$$= -2xy^2 + 2(2+x^2)^2 y^3$$

$$\therefore\ y_{i+1} = y_i - h(2+x_i^2)y_i^2 + \frac{h^2}{2}\left(-2x_i y_i^2 + 2(2+x_i^2)y_i^3\right)$$

$y_0 = 1.$ True

$y_1 = .82$ $.833$

$y_2 = .695$ $.712$

Error is about .01 for first two terms.

(b) $y'' - 4y = \frac{1}{2}x$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - 4y_i = \tfrac{1}{2}x_i \qquad i = 1, \ldots, n-1$$

$$y_{i+1} - 2y_i + y_{i-1} - 4h^2 y_i = \tfrac{1}{2}x_i h^2$$

or $y_{i+1} - 4h^2 y_i - 2y_i + y_{i-1} = \frac{1}{2}x_i h^2$

$$y_{i+1} - (4h^2+2)y_i + y_{i-1} = \tfrac{1}{2}x_i h^2$$

$i=1$ $\quad -(4h^2+2)y_1 + y_2 = \frac{1}{2}x_1 h^2 - y_0$

$i=2$ $\quad y_1 - (4h^2+2)y_2 + y_3 = \frac{1}{2}x_2 h^2$

$i=n-1$ $\quad y_{n-1} - (4h^2+2)y_{n-1} = \frac{1}{2}x_{n-1}h^2 - y_n$

$$\begin{pmatrix} -(4h^2+2) & 1 & 0 & \cdots & 0 \\ 1 & -(4h^2+2) & 1 & 0 \cdots & 0 \\ 0 & 1 & \ddots & & \\ & & & & \\ & & & & -(4h^2+2) \end{pmatrix} \begin{pmatrix} y_1 \\ \\ \vdots \\ \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}h^2 x_1 - y_0 \\ \frac{1}{2}h_i^2 x_2 \\ \vdots \\ \\ \frac{1}{2}h^2 x_{n-1} - y_n \end{pmatrix}$$

$y_0 = 1, \quad y_4 = 1 \quad h = 0.25$

$$A = \begin{pmatrix} -2.25 & 1 & 0 \\ 1 & -2.25 & 1 \\ 0 & 1 & -2.25 \end{pmatrix}, \quad B = \begin{pmatrix} -.992188 \\ 0.015625 \\ -.976563 \end{pmatrix}$$

Solve $Ay = B$, $\quad y_1 = .721584$

$y_2 = 0.631378$

$y_3 = 0.71464$