Class Test Solutions Spring 2015

(1) $(1+x)y'' + xy' - y = 0$ Divide by x

$$y'' + \frac{x}{1+x}y' - \frac{1}{1+x}y = 0$$

$$p(x) = \frac{x}{1+x} \qquad e^{-\int p(x)dx} = e^{-x+\ln x}$$
$$= e^{-x}(1+x)$$

$$\int \frac{x}{1+x} = \int \frac{x+1-1}{x+1} = \int 1 - \frac{1}{1+x}$$
$$= x - \ln(1+x)$$

$$\text{So} \quad y_2 = y_1 \int \frac{e^{-x}(1+x)}{x^2}\,dx$$

$$\text{Now} \quad \frac{d}{dx}\left(\frac{e^{-x}}{x}\right) = -\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} = -\frac{e^{-x}(1+x)}{x^2}$$

$$\text{So} \quad y_2 = x\int \frac{e^{-x}(1+x)}{x^2}\,dx$$
$$= x\left(-\frac{e^{-x}}{x}\right) = -e^{-x}$$

Second solution is $y = e^{-x}$. (Any constant multiple of $-e^{-x}$ will do)

(b) $y'' + y = \cot x$. $\quad y_1 = \cos x$, $y_2 = \sin x$ are solutions of $y'' + y = 0$

$R(x) = \cot x$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$y_p = uy_1 + vy_2$

$$u = -\int \frac{y_2 R}{W}\,dx = -\int \sin x \cot x\,dx = -\int \cos x\,dx$$
$$= -\sin x$$

$$v = \int \frac{y_1 R}{W}\,dx = \int \frac{\cos x \cos x}{\sin x}\,dx$$

$$= \int \frac{\cos^2 x}{\sin x}\,dx = \int \frac{1-\sin^2 x}{\sin x}\,dx$$

$$= \int (\operatorname{cosec} x - \sin x)\,dx$$

$$= -\ln(\operatorname{cosec} x + \cot x) + \cos x$$

$$\therefore\ y_p = -\sin x \cos x + \sin x\left(-\ln(\operatorname{cosec} x + \cot x) + \cos x\right)$$

$$= -\ln(\operatorname{cosec} x + \cot x)\sin x$$

(b) $y'' - 4xy' + 2y = 0$. Put $y = \sum_{n=0}^{\infty} a_n x^n$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n$$

$$= 2a_2 + 2a_0 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty}(4n-2)a_n x^n$$

$$= 2a_2 + 2a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - 4(n-2)a_n\right]x^n = 0$$

$$\therefore\ a_2 = -a_0 \qquad a_{n+2} = \frac{(4n-2)a_n}{(n+2)(n+1)} \quad n \geq 1$$

$$a_{n+2} = \frac{2(2n-1)a_n}{(n+2)(n+1)}. \quad a_4 = \frac{6}{4\times3}a_2 = -\frac{6}{4\times3}a_0 = -\frac{2\times3}{4\times3}a_0$$

$$a_6 = \frac{2\times7}{6\times5}a_4 = \frac{-2^2(3\times7)}{3\times4\times5\times7}a_0,\ a_8 = \frac{-2^3\times(11\times7\times3)}{3\times4\times5\times6\times7\times8}a_0$$

$$a_8 = -\frac{2^4(3\times7\times11)}{2\times3\times4\times\ldots\times8}a_0$$

$$\therefore\ a_{2n} = -\frac{2^{n+1}(3\times7\times\ldots(4n-5))}{(2n)!}a_0$$

To get $4n-5$, write term as $an+b$. Then $3a+b=7$, $4a+b=11$. Solving gives result

$n=1,\quad a_3 = \frac{4-2}{3\times 2} a_1 = \frac{2}{3\times 2} a_1$

$a_5 = \frac{12-2}{5\times 4} a_3 = \frac{2\times 10}{5!} a_1$

$a_7 = \frac{20-2}{7\times 6} a_5 = \frac{2\times 10\times 18}{7!} a_1, \quad a_9 = \frac{2\times 10\times 18\times 26}{9!} a_1$

So $a_{2k+1} = \frac{2\times 10\times \cdots (8k-6)}{(2k+1)!} a_1$

To get $8k-6$ note the general term is $ak+b$

So $k=2$ gives $2a+b=10$

$k=3$ gives $3a+b=18$

solving gives $a=8,\ b=-6$

$$\therefore\ y = a_0\left(1 - \sum_{k=1}^{\infty} \frac{2^{k+1}(3\times 7\times \cdots (4k-5))}{(2k)!} x^{2k}\right)$$

$$+\ a_1 \sum_{k=0}^{\infty} \frac{2\times 10\times \cdots (8k-6)}{(2k+1)!} x^{2k+1}$$

or $\sum_{k=0}^{\infty} \frac{2\times 10\times 18\times \cdots}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{2^k (1\times 5\times 9\times \cdots (4k-3))}{(2k+1)!} x^{2k+1}$

(3) $y'' - 4y = \sin t,\quad y(0)=0,\quad y'(0)=2$

$\mathcal{L}(y'' - 4y) = s^2 Y(s) - s y(0) - y'(0) - 4Y(s)$

$\boxed{\mathcal{L}(y) = Y(s)}$ $\quad = (s^2-4)Y(s) - 2 = \mathcal{L}(\sin t)$

$= \frac{1}{s^2+1}$

~~$Y(s) =$~~

$\therefore\ (s^2-4)Y(s) = \frac{1}{s^2+1} + 2$

$Y(s) = \frac{2}{s^2-4} + \frac{1}{(s^2+1)(s^2-4)}$

$= \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s+2}\right] + \frac{1}{(s^2+1)(s^2-4)}$

$\frac{1}{(s^2+1)(s^2-4)} = \frac{as+b}{s^2+1} + \frac{C}{s-2} + \frac{D}{s+2}$

Multiply both sides by $s-2$

$$C + \frac{D(s-2)}{s+2} + \frac{(as+b)(s-2)}{s^2+1} = \frac{1}{(s^2+1)(s+2)}$$

$s=2$ gives $C = \dfrac{1}{4\times 5} = \dfrac{1}{20}$

Then $D + \dfrac{C(s+2)}{s-2} + \dfrac{(as+b)(s+2)}{s^2+1} = \dfrac{1}{(s^2+1)(s-2)}$

$s=-2$ gives

$$D = \frac{-1}{20}$$

Now

$$\frac{1}{(s^2+1)(s^2-4)} = \frac{as+b}{s^2+1} + \frac{1}{20}\left[\frac{1}{s-2} - \frac{1}{s+2}\right]$$

$s=0$ gives

$$\frac{-1}{4} = b + \frac{1}{20}\left[-\frac{1}{2} - \frac{1}{2}\right], \quad b = \frac{1}{20} - \frac{1}{4} = \frac{-1}{5}$$

Finally

$$\frac{1}{(s^2+1)(s^2-4)} = -\frac{1}{5}\,\frac{1}{s^2+1} + \frac{1}{20}\left[\frac{1}{s-2} - \frac{1}{s+2}\right] + \frac{as}{s^2+1}$$

Take $s=1$

$$\frac{a}{2} - \frac{1}{10} + \frac{1}{20}\left[-\frac{1}{1} - \frac{1}{3}\right] = \frac{1}{2\times(-3)}$$

or $\dfrac{a}{2} - \dfrac{1}{6} = \dfrac{-1}{6}$ $\therefore$ $a = 0$

So

$$Y(s) = \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s+2}\right] - \frac{1}{5}\,\frac{1}{s^2+1} + \frac{1}{20}\left[\frac{1}{s-2} - \frac{1}{s+2}\right]$$

$$y(t) = \frac{1}{2}\left(e^{2t} - e^{-2t}\right) - \frac{1}{5}\sin t + \frac{1}{20}e^{2t} - \frac{1}{20}e^{-2t}$$

$$= \frac{11}{20}e^{2t} - \frac{11}{20}e^{-2t} - \frac{\sin t}{5}$$