Question 1 (8 + 10 + 7 = 25 marks).

(a) The ordinary differential equation

$$x^2(x+1)y'' - xy' + y = 0,$$

has a solution $y_1(x) = x$. Use this to find a second linearly independent solution y_2.

(b) Use Variation of Parameters to find the general solution of the ODE

$$y'' + y = \tan(x).$$

You may need $\int \sec(x)dx = \ln(\sec(x) + \tan(x))$.

(c) Solve the equation

$$x^2y'' + 4xy' + (2 + 4x^6)y = 0,$$

in terms of Bessel functions.

Question 2 (12 + 13 = 25 marks).

(a) For the ODE

$$(1-x^2)y'' - 2xy' + 12y = 0.$$

obtain a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Determine the radius of convergence of your series solutions.

(b) Consider the ODE $3xy'' + y' - y = 0$.

(i) Look for a solution of the form $y = x^s \sum_{n=0}^{\infty} a_n x^n$. Show that we must have $s = 0$ or $s = 2/3$.

(ii) Show that the coefficients satisfy

$$a_{n+1} = \frac{a_n}{(n+s+1)(3n+3s+1)}, \quad n = 0, 1, 2, 3, ...$$

(iii) Use the recurrence relation for a_n to generate two linearly independent solutions of the equation.

Question 3 (12 + 7 + 6= 25 marks).

(a) Use the Laplace transform to solve the ODE

$$y'' - 4y = \cos t, \;\; y(0) = 0, \;\; y'(0) = 1.$$

(b) Let $F(s)$ be the Laplace transform of $f(t)$. Obtain an expression for the Laplace transform of $e^{bt}H(t-a)f(t-a)$ in terms of F. Here $H(t-a)$ is the Heaviside step function.

(c) Explain why the function

$$F(s) = e^{-4s}\left(\frac{s^2 - s + 1}{2s^6 + 5s^3 + 2s + 5}\right),$$

is a Laplace transform. (DO NOT TRY TO INVERT THIS LAPLACE TRANSFORM).

Question 4 (13 + 12 = 25 marks).

(a) Consider the following initial and boundary value problem for the heat equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ 0 \le x \le 1, \ t > 0,$$
$$u(0,t) = u(1,t) = 0, \ u(x,0) = x - x^2.$$

(i) By looking for solutions of the heat equation of the form $u(x,t) = X(x)T(t)$ show that the functions X and T must satisfy the problems

$$X''(x) = \lambda X(x), \quad X(0) = X(1) = 0,$$

and $T'(t) = \lambda T(t)$ for some constant λ. Hence obtain expressions for X and T.

(ii) Determine the solution of the given problem for the heat equation.

(b) Set up a finite difference scheme for solving the boundary value problem

$$y'' - 4y = x, \ x \in [0,1],$$

subject to the boundary conditions $y(0) = 0, y(1) = 1$. Show that this leads to a linear system of the form $Ay = b$ where A is the tridiagonal matrix

$$A = \begin{pmatrix} -4h^2-2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -4h^2-2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -4h^2-2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & -4h^2-2 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}, \quad b = \begin{pmatrix} h^2 x_1 \\ \vdots \\ h^2 x_{n-1} - 1 \end{pmatrix}.$$

Solve the resulting system in the case when $n = 4$. i.e $h = 0.25$. You may need the approximation

$$y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \ y_i = y(x_i).$$

Table of integrals

$$\int u^n\,du = \frac{u^{n+1}}{n+1}$$

$$\int \frac{du}{u} = \log|u|$$

$$\int e^u\,du = e^u$$

$$\int \cos u\,du = \sin u$$

$$\int \sin u\,du = -\cos u$$

$$\int \text{cosech}^2 u\,du = -\coth u$$

$$\int \tan^2 u\,du = u - \tan u$$

$$\int \sec u \tan u\,du = \sec u$$

$$\int \csc u \cot u\,du = -\csc u$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$$

$$\int u^n \ln u du = \frac{u^{1+n}}{1+n}\ln u - \frac{u^{1+n}}{(1+n)^2}$$

$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u$$

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\frac{u}{a} = \log\left(u+\sqrt{a^2+u^2}\right)$$

$$\int \frac{du}{1-u^2} = \tanh^{-1} u = \frac{1}{2}\log\left|\frac{1+u}{1-u}\right|$$

$$\int \cosh u\,du = \sinh u$$

$$\int \sinh u\,du = \cosh u$$

$$\int \tanh u\,du = \log\cosh u$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln u du = u\ln u - u$$

$$\int u\sin(au)du = \frac{\sin(au)}{a^2} - \frac{u\cos(au)}{a}$$

$$\int u\cos(au)du = \frac{\cos(au)}{a^2} + \frac{u\sin(au)}{a}$$

$$\int u^2\sin(au)du = \frac{2u\sin(au)}{a^2} - \frac{(a^2u^2-2)\cos(au)}{a^3}$$

$$\int u^2\cos(au)du = \frac{2u\cos(au)}{a^2} + \frac{(a^2u^2-2)\sin(au)}{a^3}.$$

Table of Laplace transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$
$$\mathcal{L}(e^{-at}) = \frac{1}{s+a}$$
$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$$
$$\mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2}$$
$$\mathcal{L}(J_0(t)) = \frac{1}{\sqrt{1+s^2}}.$$

Variation of parameters

Given that y_1 and y_2 are solutions of the ODE

$$y''(x) + b(x)y'(x) + c(x)y(x) = 0,$$

we seek a particular solution of the ODE

$$y''(x) + b(x)y'(x) + c(x)y(x) = f(x),$$

by looking for solutions of the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$. The functions u and v must satisfy

$$u'y_1 + v'y_2 = 0,$$
$$u'y_1' + v'y_2' = f(x).$$

Bessel Functions

Bessel's differential equation $t^2u'' + tu' + (t^2 - \alpha^2)u = 0$ may be transformed into the equation

$$x^2y'' + (1-2s)xy' + ((s^2 - r^2\alpha^2) + a^2r^2x^{2r})y = 0$$

under the change of variables $t = ax^r$ and $y(x) = x^s u(t)$.

$$a_0 = \frac{1}{L}\int_0^L f(x)dx,$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$