DEs Class Test Solutions 2016

1) $x^2y'' - 4xy' + 4y = 0 \qquad y_1 = x$

Divide by x^2 $\quad y'' - \frac{4}{x}y' + \frac{4}{x^2}y = 0$

$p(x) = -\frac{4}{x}. \quad y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2}\,dx$

$= x\int \frac{e^{\int \frac{4}{x}}}{x^2}\,dx = x\int \frac{x^4}{x^2}\,dx = x\int x^2\,dx$

$= \frac{1}{3}x^4$

$\left[e^{\int \frac{4}{x}} = e^{4\ln x} = e^{\ln x^4} = x^4 \right]$

Take $y_2 = x^4$

Now solve $y'' - \frac{4}{x}y' + \frac{4}{x^2}y = x$

$R(x) = x, \quad W(y_1, y_2) = \begin{vmatrix} x & x^4 \\ 1 & 4x^3 \end{vmatrix} = 4x^4 - x^4 = 3x^4$

$u = \int \frac{-y_2R}{W} = -\int \frac{x^4 x}{3x^4}\,dx = -\frac{1}{3}\int x\,dx = -\frac{1}{6}x^2$

$v = \int \frac{y_1R}{W} = \int \frac{x\,x}{3x^4} = \frac{1}{3}\int \frac{dx}{x^2} = -\frac{1}{3x}$

So $\quad y_p = uy_1 + vy_2 = -\frac{1}{6}x^3 - \frac{1}{3}x^3 = -\frac{1}{2}x^3$

$\therefore \quad y = C_1x + C_2x^4 - \frac{x^3}{2}$

(2) $y'' + (1-x)y' + y = 0. \quad y = \sum_{n=0}^{\infty} a_n x^n$

$\therefore \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + (1-x)\sum_{n=1}^{\infty} na_nx^{n-1} + \sum_{n=0}^{\infty} a_nx^n$

$= \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=1}^{\infty} na_nx^n + \sum_{n=0}^{\infty} a_nx^n$

$$= 2a_2 + a_1 + a_0 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n$$
$$+ \sum_{n=1}^{\infty} a_n x^n$$

$$= 2a_2 + a_1 + a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + (-n+1)a_n\right]x^n = 0$$

$$\therefore\ 2a_2 + a_1 + a_0 = 0$$

and $(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + (1-n)a_n = 0$

So

$$a_2 = -\frac{1}{2}(a_1 + a_0)$$

$$a_{n+2} = \frac{-\left[(n+1)a_{n+1} + (1-n)a_n\right]}{(n+2)(n+1)}$$

$n = 1$

$$a_3 = -\frac{\left[2a_2 + (1-1)a_1\right]}{6} = \frac{(-1)^2(a_1 + a_0)}{6}$$
$$= \frac{a_1 + a_0}{6}$$

$$a_4 = -\frac{1}{12}\left[3a_3 - a_2\right] = -\frac{1}{12}\left[\frac{a_1 + a_0}{2} + \frac{1}{2}(a_1 + a_0)\right]$$
$$= -\frac{1}{12}(a_1 + a_0)$$

$$a_5 = -\frac{1}{20}\left(4a_4 + (1-3)a_3\right)$$
$$= -\frac{1}{20}\left[-\frac{1}{3}(a_1 + a_0) - 2\frac{(a_0 + a_1)}{6}\right]$$
$$= \frac{a_0 + a_1}{30} \quad \text{etc.}$$

So $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$$= a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{30}x^5 \cdots\right)$$
$$+ a_1\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{30}x^5 \cdots\right)$$

3 (a) $$\frac{s+2}{(s+4)(s^2+16)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+16}.$$

So $$A = \frac{(s+2)}{s^2+16} - \frac{(Bs+C)(s+4)}{s^2+16}. \quad \text{put } s=-4$$

$$A = \frac{-2}{32} = -\frac{1}{16}$$

$$\therefore \frac{s+2}{(s+4)(s^2+16)} = \frac{-1}{16(s+4)} + \frac{Bs+C}{s^2+16} \quad \text{put } s=0$$

$$\frac{2}{4(16)} = -\frac{1}{16(4)} + \frac{C}{16} \quad \therefore C = 3/4$$

$$\therefore \frac{s+2}{(s+4)(s^2+16)} = \frac{-1}{16(s+4)} + \frac{3}{4(s^2+16)} + \frac{Bs}{s^2+16}. \quad \text{put } s=1$$

$$\frac{B}{17} = \frac{3}{5(17)} + \frac{1}{16(5)} - \frac{3}{4(17)}.$$

$$\therefore B = \frac{1}{16}.$$

and $$\frac{s+2}{(s+4)(s^2+16)} = -\frac{1}{16(s+4)} + \frac{3}{4(s^2+16)} + \frac{s}{16(s^2+16)}$$

and $$\mathcal{L}^{-1}\left(\frac{s+2}{(s+4)(s^2+16)}\right) = -\frac{1}{16}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) + \frac{3}{16}\mathcal{L}^{-1}\left(\frac{4}{s^2+16}\right) + \frac{1}{16}\mathcal{L}^{-1}\left(\frac{s}{s^2+16}\right)$$

$$= -\frac{1}{16}e^{-4t} + \frac{3}{16}\sin(4t) + \frac{1}{16}\cos(4t)$$

(b) $$\mathcal{L}(h) = \int_0^\infty e^{-t} t f(t) e^{-st}\,dt = \int_0^\infty t f(t) e^{-(s+1)t}\,dt$$

$$= -\int_0^\infty f(t)\left(\frac{d}{ds}e^{-(s+1)t}\right)dt = -\frac{d}{ds}\int_0^\infty f(t)e^{-(s+1)t}\,dt$$

$$= -\frac{d}{ds}F(s+1).$$