2016 Exam. Solutions to Q2 (b)

We solve $3xy'' + y' - y = 0$. Note the solution can be expressed in terms of Bessel functions. Try this as an exercise. Here we use Frobenius.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+s}$. Then substitution into the DE we have

$$\sum_{n=0}^{\infty} 3(n+s)(n+s-1)a_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} - \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$(3s(s-1)+s)a_0 x^s + \sum_{n=1}^{\infty}[3(n+s)(n+s-1)+(n+s)a_n x^{n+s}]a_n x^{n+s-1}$$

$$-\sum_{n=0}^{\infty} a_n x^{n+s} = s(3s-2)a_0 x^s +$$

$$\sum_{n=0}^{\infty}[(n+s+1)(3(n+s)+1)a_{n+1}x^{n+s} - a_n]x^{n+s} = 0.$$

So we require $s(3s-2) = 0$, giving $s = 0, s = 2/3$.

The coefficients must satisfy

$$a_{n+1} = \frac{a_n}{(n+s+1)(3n+3s+1)}.$$

Take $s = 0$. This gives

$$a_{n+1} = \frac{a_n}{(n+1)(3n+1)}.$$

Taking $n = 0$ gives $a_1 = a_0$. $n = 1$ gives $a_2 = \frac{a_1}{2.4} = \frac{a_0}{1.2.4}$. Then $a_3 = \frac{a_2}{3.7} = \frac{a_0}{1.2.3.4.7}$. $a_4 = \frac{a_0}{4!4.7.10}$ etc. In general

$$a_n = \frac{a_0}{n!(1 \times 4 \times \cdots (3n-2))}$$

for $n \geq 1$. The solution corresponding to this value of s is then

$$y_1 = a_0\left(1 + \sum_{n=1}^{\infty}\frac{x^n}{n!(1 \times 4 \times \cdots (3n-2))}\right).$$

For $s = 2/3$ we have

$$a_{n+1} = \frac{a_n}{(n+1)(3n+5)}.$$

Then $a_1 = \frac{a_0}{5}$, $a_2 = \frac{a_1}{2.8} = \frac{a_0}{2.5.8}$ $a_3 = \frac{a_2}{3.11} = \frac{a_0}{3!(5.8.11)}$. In general

$$a_n = \frac{a_0}{n!(1 \times 5 \times \cdots \times (3n+2)}.$$

The solution is

$$y_2 = a_0 x^{2/3}\sum_{n=0}^{\infty}\frac{x^n}{n!(1 \times 5 \times \cdots \times (3n+2)}.$$

Numerical Methods problem solutions

Question 1.

(a) We have the equation $y' = x^2 y(x)^2$ with the initial condition $y(0) = 1$. We work on the interval $[0, 1]$. So $y_0 = 1$. Now

$$\begin{aligned}\frac{d}{dx}\left((x^2 y(x)^2)\right) &= 2xy^2 + x^2\frac{d}{dx}(y^2)\\ &= 2xy^2 + x^2 2yy'(x)\\ &= 2xy^2 + x^2 2yx^2y^2 = 2xy^2 + 2x^4y^3,\end{aligned}$$

since $y' = x^2y^2$. Thus the second order Taylor scheme will be

$$y_{i+1} = y_i + hx_i^2y_i^2 + h^2\left(x_iy_i^2 + x_i^4y_i^3\right).$$

(b) We have the Riccati equation $y' = x^2 + y^2$, $y(0) = 1$. So $y_0 = 1$. Now $y'' = \frac{d}{dx}(x^2 + y^2)$