SEAT NUMBER:

STUDENT NUMBER:

SURNAME:
(FAMILY NAME)

OTHER NAMES:

**This paper and all materials issued must be returned at the end of the examination.
They are not to be removed from the exam centre.**

Examination Conditions:

It is your responsibility to fill out and complete your details in the space provided on all the examination material provided to you. Use the time before your examination to do so as you will not be allowed any extra time once the exam has ended.

You are **not** permitted to have on your desk or on your person any unauthorised material. This includes but not limited to:

- Mobile phones
- Smart watches and bands
- Electronic devices
- Draft paper (unless provided)
- Textbooks (unless specified)
- Notes (unless specified)

You are **not** permitted to obtain assistance by improper means or ask for help from or give help to any other person.

If you wish to **leave and be re-admitted** (including to use the toilet), you have to wait until **90 mins** has elapsed.

If you wish to **leave the exam room permanently**, you have to wait until **60 mins** has elapsed.

You are not permitted to leave your seat (including to use the toilet) during the final 15 mins.

During the examination **you must first seek permission** (by raising your hand) from a supervisor before:

- Leaving early
- Using the toilet
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Misconduct action will be taken against you if you breach university rules.

Declaration: I declare that I have read the advice above on examination conduct and listened to the examination supervisor's instructions for this exam. In addition, I am aware of the university's rules regarding misconduct during examinations. I am not in possession of, nor do I have access to, any unauthorised material during this examination. I agree to be bound by the university's rules, codes of conduct, and other policies relating to examinations.

Signature:

Date:

37335 Differential Equations

Time Allowed: 120 minutes.

Reading time: 10 minutes.

Reading time is for reading only. You are not permitted to write, calculate or mark your paper in any way during reading time.

Closed Book
Non-programmable Calculators Only

Permitted materials for this exam:

None

Materials provided for this exam:

1 x 5 Page Booklet
1 x 20 Page Booklet
1 Table of Integrals
1 Formula Sheet

Students please note:

Answer all questions in the 20 page booklet. The 5 page booklet is for rough working.

Do not open your exam paper until instructed.

Rough work space

Do **not** write your answers on this page.

Question 1 (8 + 10 + 7 = 25 marks).

(a) The ordinary differential equation

$$x(1-x)y'' + 3xy' - 3y = 0,$$

has a solution $y_1(x) = x$. Use this to find a second linearly independent solution y_2.

(b) Use Variation of Parameters to find the general solution of the ODE

$$x^2y'' + 2xy' - 6y = x\ln x.$$

The equation $x^2y'' + 2xy' - 6y = 0$ has solutions of the form $y = x^a$. Substitution into the equation will produce a quadratic satisfied by a, with two distinct roots.

(c) Solve the equation

$$x^2y'' + 9xy' - \left(9 + 4x^2\right)y = 0$$

in terms of Bessel functions.

Question 2 (12 + 13 = 25 marks).

(a) For the ODE

$$(1-x^2)y'' - 2xy' + 12y = 0,$$

obtain two linearly independent solutions by letting $y = \sum_{n=0}^{\infty} a_n x^n$.

(b) Consider the ODE

$$2xy'' + (3-x)y' + 6y = 0.$$

(i) Look for a solution of the form $y = x^s \sum_{n=0}^{\infty} a_n x^n$. Show that $s = 0$ or $s = -1/2$.

(ii) Show that the coefficients satisfy

$$a_n = \frac{(n+s-7)a_{n-1}}{(n+s)(2n+2s+1)}, \quad n = 1, 2, 3, ...$$

(iii) Use the recurrence relation for a_n to generate two linearly independent solutions of the equation. Show that one is a polynomial of degree 6.

Question 3 (12 + 7 + 6= 25 marks).

(a) Use the Laplace transform to solve the ODE

$$y'' + 4y = \sin(3t),\ y'(0) = 0,\ y'(0) = 0.$$

(b) Let $F(s)$ be the Laplace transform of $f(t)$. Obtain an expression for the Laplace transform of $e^{-bt}f'(at)$ in terms of F and $f(0)$.

(c) Invert the Laplace transform

$$F(s) = \frac{1}{s^2}e^{-1/s},$$

as an infinite series.

Question 4 (13 + 12 = 25 marks).

(a) Consider the following initial and boundary value problem for the wave equation.

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1,\ t > 0,$$
$$u(0,t) = u(1,t) = 0,\ u(x,0) = 0, u_t(x,0) = x^2 - x.$$

(i) By looking for solutions of the wave equation of the form $u(x,t) = X(x)T(t)$ show that the functions X and T must satisfy the problems

$$X''(x) = \lambda X(x), \quad X(0) = X(1) = 0,$$

and $T''(t) = \lambda T(t)$ for some constant λ. Determine the values of l which lead to a nonzero solution and hence obtain expressions for X and T.

(ii) Determine the solution of the given problem for the wave equation.

(b) Set up a finite difference scheme for solving the boundary value problem

$$y'' + 9y = f(x),\ x \in [0,1],$$

subject to the boundary conditions $y(0) = 2, y(1) = 1$. Here f is a continuous function. Show that this leads to a linear system of the form $Ay = b$ where A is the tridiagonal matrix

$$A = \begin{pmatrix} 9h^2-2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 9h^2-2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 9h^2-2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & 9h^2-2 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}, \quad b = \begin{pmatrix} h^2 f(x_1) - 2 \\ \vdots \\ h^2 f(x_{n-1}) - 1 \end{pmatrix}.$$

Solve the resulting system in the case when $f(x) = \frac{1}{2}x$ and $n = 4$. i.e $h = 0.25$. You may need the approximation

$$y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2},\ y_i = y(x_i).$$

Table of integrals

$$\int u^n\,du = \frac{u^{n+1}}{n+1}$$

$$\int \frac{du}{u} = \log|u|$$

$$\int e^u\,du = e^u$$

$$\int \cos u\,du = \sin u$$

$$\int \sin u\,du = -\cos u$$

$$\int \operatorname{cosech}^2 u\,du = -\coth u$$

$$\int \tan^2 u\,du = u - \tan u$$

$$\int \sec u \tan u\,du = \sec u$$

$$\int \csc u \cot u\,du = -\csc u$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$$

$$\int u^n \ln u du = \frac{u^{1+n}}{1+n}\ln u - \frac{u^{1+n}}{(1+n)^2}$$

$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u$$

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\frac{u}{a} = \log\left(u+\sqrt{a^2+u^2}\right)$$

$$\int \frac{du}{1-u^2} = \tanh^{-1} u = \frac{1}{2}\log\left|\frac{1+u}{1-u}\right|$$

$$\int \cosh u\,du = \sinh u$$

$$\int \sinh u\,du = \cosh u$$

$$\int \tanh u\,du = \log\cosh u$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln u du = u\ln u - u$$

$$\int u\sin(au)du = \frac{\sin(au)}{a^2} - \frac{u\cos(au)}{a}$$

$$\int u\cos(au)du = \frac{\cos(au)}{a^2} + \frac{u\sin(au)}{a}$$

$$\int u^2\sin(au)du = \frac{2u\sin(au)}{a^2} - \frac{(a^2u^2-2)\cos(au)}{a^3}$$

$$\int u^2\cos(au)du = \frac{2u\cos(au)}{a^2} + \frac{(a^2u^2-2)\sin(au)}{a^3}.$$

Table of Laplace transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$
$$\mathcal{L}(e^{-at}) = \frac{1}{s+a}$$
$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$$
$$\mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2}$$
$$\mathcal{L}(J_0(t)) = \frac{1}{\sqrt{1+s^2}}.$$

Variation of parameters

Given that y_1 and y_2 are solutions of the ODE

$$y''(x) + b(x)y'(x) + c(x)y(x) = 0,$$

we seek a particular solution of the ODE

$$y''(x) + b(x)y'(x) + c(x)y(x) = f(x),$$

by looking for solutions of the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$. The functions u and v must satisfy

$$u'y_1 + v'y_2 = 0, \; u'y_1' + v'y_2' = f(x).$$

Bessel Functions

Bessel's differential equation $t^2u'' + tu' + (t^2 - \alpha^2)u = 0$ may be transformed into the equation

$$x^2y'' + (1-2s)xy' + ((s^2 - r^2\alpha^2) + a^2r^2x^{2r})y = 0$$

under the change of variables $t = ax^r$ and $y(x) = x^s u(t)$.

Fourier Coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)dx,$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$